Unsteady State Heat Conduction in a Bounded Solid

Consider a sphere of radius $R$. Initially the sphere is at a uniform temperature $T_0$. It is cooled by convection to an air stream at temperature $T_a$. What do we do to describe the conduction and the temperature distribution within the sphere?

You will recall that in the last lecture the uniform temperature approximation was quite unrealistic when it came to the time scale for cooling.

There are a number of methods that we can use to attack this problem. The first is to do a shell balance on a differential shell within the sphere. That exercise leads to

$$4\pi[r^2q_r]- 4\pi[r^2q_r]_{r+\Delta r} = \frac{\partial}{\partial t}(\rho U 4\pi r^2 \Delta r)$$

If we divide both sides by $4\pi r^2 \Delta r$, we obtain

$$\frac{[r^2q_r]_r-[r^2q_r]_{r+\Delta r}}{\Delta r} = r^2 \frac{\partial}{\partial t}(\rho U)$$

Recognizing that for a solid, $C_p \approx C_v$ and if the density is constant, we observe that

$$-\frac{1}{r^2} \frac{\partial}{\partial r}[r^2q_r] \rho \frac{\partial}{\partial t}(\rho U) = \rho \left(\frac{\partial U}{\partial T}\right)_{\rho} \frac{\partial T}{\partial t} = \rho C_p \frac{\partial T}{\partial t}$$

so that

$$-\frac{1}{r^2} \frac{\partial}{\partial r}[r^2q_r] = \rho C_p \frac{\partial T}{\partial t}$$

At this point we should introduce Fourier’s law of heat conduction
The time-dependent equation for heat conduction becomes

$$\rho C_p \frac{\partial T}{\partial t} = \frac{1}{r^2 \frac{\partial}{\partial r}} \left[ r^2 \frac{k_s}{\rho C_p} \frac{\partial T}{\partial r} \right]$$

We can define the **Fourier Diffusivity or thermal diffusivity** as

$$\alpha = \frac{k_s}{\rho C_p}$$

The conduction equation becomes mathematically equivalent to the diffusion equation

$$\frac{\partial T}{\partial t} = \alpha \frac{1}{r^2 \frac{\partial}{\partial r}} \left[ r^2 \frac{\partial T}{\partial r} \right]$$

**Boundary and Initial Conditions**

The problem becomes defined only when the initial conditions and the boundary conditions are specified.

For t < 0, we have T = T₀ in \( r \in (0, R) \)

For r = 0, the temperature field is bounded for all t > 0

or \( \frac{\partial T}{\partial r} = 0 \) (symmetry)

For r = R, we have **Newton’s Law of Cooling**, that is

$$q_r = -k_s \frac{\partial T}{\partial r} = h(T - T_a)$$
The Dimensionless Description

My passion for a dimensionless problem exists because I’m fundamentally lazy and I don’t want to solve a problem each time I formulate it.

Let’s put the equations in dimensionless form

$$\theta = \frac{T - T_a}{T_0 - T_a}; \quad \xi = \frac{r}{R}$$

$$X_{Fo} = \frac{\alpha t}{R^2}; \quad Bi = \frac{hR}{k_s}$$

Applying these definitions to the dimensionless conduction equation and its boundary and initial conditions, we obtain

$$\frac{\partial \theta}{\partial x_{Fo}} = \frac{1}{\xi^2} \frac{\partial}{\partial \xi} \left( \xi^2 \frac{\partial \theta}{\partial \xi} \right)$$

$$\theta = 1 \text{ at } x_{Fo} < 0; \quad \frac{\partial \theta}{\partial \xi} = 0 \text{ at } \xi = 0$$

$$-\frac{\partial \theta}{\partial \xi} = Bi \theta \text{ at } \xi = 1$$

It follows that $\Theta = f(\xi, x_{Fo}, Bi)$. The solution procedure then uses a simple transformation $z = \xi \theta$ to obtain the form of the problem that we have already seen and solved in gory detail. The most common problem is when the $Bi \to \infty$, then $\Theta = f(\xi, x_{Fo})$ since the boundary condition 3 becomes $\Theta = 0$
The Adimensional Problem

Let's try to solve the problem by first making the equation adimensional. Define the following:

\[ z = \xi \theta ; \quad \xi = \frac{r}{R} ; \quad X_{Fo} = \frac{\alpha t}{R^2} \]

The differential equation and the initial and boundary conditions become:

\[ \frac{\partial z}{\partial X_{Fo}} - \frac{\partial^2 z}{\partial \xi^2} = 0 \]

\[ z = \xi \text{ at } x_{Fo} < 0 ; \quad z = 0 \text{ at } \xi = 0 \]

\[ -\frac{\partial z}{\partial \xi} = (Bi + 1)z \text{ at } \xi = 1 \]

The equation and its conditions are dimensionless and parameter free.

Next Question

How do we solve the equation?

Suppose \( z \) has the form \( z = Y(X_{Fo})G(\xi) \)

\[ \frac{\partial z}{\partial X_{Fo}} - \frac{\partial^2 z}{\partial \xi^2} = \frac{\partial YG}{\partial X_{Fo}} - \frac{\partial^2 YG}{\partial \xi^2} = G \frac{dY}{dX_{Fo}} - Y \frac{d^2 G}{d\xi^2} = 0 \]
The equation is separable in the form

\[
\frac{1}{Y} \frac{dY}{dX_{Fo}} = \frac{1}{G} \frac{d^2G}{d\xi^2} = -\lambda^2
\]

\[
\frac{dY}{dX_{Fo}} = -\lambda^2 Y \quad ; \quad \frac{d^2G}{d\xi^2} = -\lambda^2 G
\]

Integrating each of the equations we obtain

\[
Y(X_{Fo}) = Ke^{-\lambda^2 X_{Fo}} \quad \text{and} \quad G(\xi) = A\sin(\lambda \xi) + B\cos(\lambda \xi)
\]

The solution for \( y(\theta,\eta) \) has the form

\[
z(\xi, X_{Fo}) = \left[ A\sin(\lambda \xi) + B\cos(\lambda \xi) \right] e^{-\lambda^2 X_{Fo}}
\]

We can construct the exact solution using the boundary conditions

\[
z(0, X_{Fo}) = \left[ A\sin(0) + B\cos(0) \right] e^{-\lambda^2 X_{Fo}} = 0
\]

It follows that \( B \) must be 0 if the condition is true for all \( \theta > 0 \)

Now the other boundary condition

\[
z(1, X_{Fo}) = \left[ A\sin(\lambda) \right] e^{-\lambda^2 X_{Fo}} = 0
\]

Now this is true for all \( X_{Fo} > 0 \) if and only if \( \sin(\lambda) = 0 \)

but \( \sin(\lambda) = 0 \) only where \( \lambda = n\pi \) where \( n = 0, 1, 2, \ldots \)
This means there are a countable infinity of solutions so that

\[ z(\xi, X_{Fo}) = \sum_{n=1}^{\infty} e^{-\lambda X_{Fo}} \left( A_n \sin(n\pi \xi) \right) \]

To obtain the coefficients \( A_n \), we need to use the initial condition.

\[ z = \xi \text{ at } X_{Fo} < 0 \]

\[ z(\xi, 0) = \sum_{n=1}^{\infty} \left( A_n \sin(n\pi \xi) \right) = \xi \]

To determine the coefficients, we can use the orthogonality properties of the sine and cosine functions. (See Appendix)

\[ \int_{-1}^{1} \sin(n\pi \xi) \sin(m\pi \xi) d\xi = \begin{cases} 0 & \text{for } m \neq n \\ \pi & \text{for } m = n \end{cases} \]
We integrate
\[ \int_0^1 z(\xi,0)\sin(m\pi\xi)\,d\xi = \int_0^1 \xi\sin(m\pi\xi)\,d\xi \]
\[ \sum_{n=1}^{\infty} \left( A_n \int_0^1 \sin(n\pi\xi)\sin(m\pi\xi)\,d\xi \right) = \int_0^1 \xi\sin(m\pi\xi)\,d\xi \]

You might remember that the first sine integral is non-zero if and only if \( n = m \). Now the equation for \( A_n \) is
\[ A_n = \frac{-\int_0^1 \xi\sin(n\pi\xi)\,d\xi}{\int_0^1 \sin^2(n\pi\xi)\,d\xi} = \frac{1}{n\pi} \int_0^{n\pi} x\sin(x)\,dx \]
\[ \frac{1}{n\pi} \int_0^{n\pi} \sin^2(x)\,dx \]

The result for the definite integrals follow from what I give above. It follows that
\[ A_n = \frac{4}{\pi} \frac{(1 - (-1)^m - 1)}{2} \]
so that the solution can be described as:
\[ z(\xi, X_{Fo}) = \frac{4}{\pi} \sum_{n=1}^{\infty} \left( \frac{1 - (-1)^n}{2n} e^{-n^2\pi^2 X_{Fo}} \sin(n\pi\xi) \right) \]
A LITTLE EXERCISE

Prove that the following is equivalent

\[ z(\xi, X_{Fo}) = \frac{4}{\pi} \sum_{n=0}^{\infty} \left( \frac{1}{2n + 1} e^{-(2n + 1)^2 \pi^2 X_{Fo}} \sin((2n + 1)\pi \xi) \right) \]

A Return to the Original Problem

“ How long does it take to get a fully developed temperature field in the sphere?”

Recall the initial definitions

\[ \theta = \frac{T - T_a}{T_0 - T_a} \quad ; \quad \xi = \frac{r}{R} \]

\[ X_{Fo} = \frac{\alpha t}{R^2} \quad ; \quad Bi = \frac{hR}{k_s} \]

When \( \theta = 0 \) everywhere, then the temperature profile is developed. The solution we saw was

\[ \theta(\xi, X_{Fo}) = \frac{z(\xi, X_{Fo})}{\xi} \]

\[ \theta(\xi, X_{Fo}) = \frac{4}{\xi \pi} \sum_{n=0}^{\infty} \left( \frac{1}{2n + 1} e^{-(2n + 1)^2 \pi^2 X_{Fo}} \sin((2n + 1)\pi \xi) \right) \]
Now regardless of location in the sphere, $\theta$ is small when $\xi$ is large. Now how large? The first term in the series is when $n = 1$, the second is when $n = 3$, etc. The first term is much greater than the subsequent one so that

$$\text{so as } \theta \to \infty \quad y(\eta, \theta) \to \frac{4}{\pi}e^{-\pi^2\theta}\sin(\pi\eta)$$

then $X_{Fo} \to \infty$ $\frac{z(\xi, X_{Fo})}{z(\xi, 0)} \to e^{-\pi X_{Fo}}$

These solutions can be compared graphically.

![Temperature Response](image)

It is apparent that by the time $X_{Fo} = 0.5$, $\theta = 0.01$, that is 1% of the original value. I’ll presume this is long enough.
You should notice though that on a semi-log plot the data are almost a straight line. What does this suggest?
Appendix
Problem --- evaluate $A_n$

The integrals in the definition of $A_n$ can be proven simply by noting that
\[ e^{ix} = \cos(x) + i \sin(x) \]

and using it to prove a number of identities involving sines and cosines.

The result is a simple relation for the Fourier coefficients. The results are

\[ \int_0^\pi \cos(nx)\sin(mx)dx = 0 \text{ for all } m, n \]

\[ \int_0^\pi \cos(nx)\cos(mx)dx = \begin{cases} 
0 & \text{for } m \neq n \\
\frac{\pi}{2} & \text{for } m = n > 0 \\
\pi & \text{for } m = n = 0 
\end{cases} \]

\[ \int_0^\pi \sin(nx)\sin(mx)dx = \begin{cases} 
\frac{\pi}{2} & \text{for } m = n > 0 \\
0 & \text{for } m \neq n 
\end{cases} \]

\[ \int_0^\pi x\sin(mx)dx = \frac{\pi}{m}(-1)^m \]

It shows that $\sin(x)$ and $\cos(x)$ are orthogonal functions. Some algebra will get you the result for $A_n$.

\[ A_n = \frac{4}{\pi} \frac{(1 - (-1)^m - 1)}{2} \]