Applications of the Energy Equation

Solids with a Uniform Temperature

Suppose a metal sphere of uniform temperature, $T$. Heat is transferred by convection with a heat transfer coefficient, $h$. The temperature of the surroundings is $T_a$.

Therefore the energy balance is

$$\rho C_p V \frac{dT}{dt} = -hA(T - T_a)$$

(Please note the difference between this equation and 11.1.1 in the text. This is the correct form)

If the sphere is initially at $T_0$, how does one describe the cooling of the sphere?

The equation is separable so that

$$\frac{dT}{T - T_a} = -\frac{hA}{\rho C_p V} dt$$

It follows that the solution is

$$\ln\frac{T - T_a}{T_0 - T_a} = -\frac{hA}{\rho C_p V} t$$

Or more explicitly

$$\frac{T - T_a}{T_0 - T_a} = e^{-\frac{hA}{\rho C_p V} t}$$
Adimensionalization

If I had defined the following:

\[ \theta = \frac{T - T_a}{T_0 - T_a} \quad \text{and} \quad \tau = \frac{\rho C_p V}{hA} \]

The solution has a simple expression ... \( \Theta = e^{-t/\tau} \)

Measurement of a Convective Heat Transfer Coefficient

Suppose a sphere of radius R in a stagnant gas of infinite extent. The heat flux from the sphere through the gas is given by Fourier’s law

\[ q_r = -k_g \frac{dT}{dr} \]

The heat flow through a spherical shell is constant so

\[ r^2q_r = -r^2k_g \frac{dT}{dr} = C \]

The boundary conditions are \( T = T_s \) at \( r = R \) and \( T \to T_a \) at \( r \to \infty \)

The solution becomes

\[
\frac{T - T_a}{T_R - T_a} = \frac{R}{r}
\]
We can calculate the heat flux at the surface as

\[ q_r \bigg|_{r = R} = -k \left( \frac{dT}{dr} \right) \bigg|_{r = R} = k \frac{(T_R - T_a)}{R} \]

We can define a heat transfer coefficient as

\[ h \equiv \frac{q_r}{T_R - T_a} = \frac{k_g}{R} \]

The corresponding Nusselt Number for heat transfer in a stagnant gas is

\[ Nu = \frac{hD}{k_g} = 2 \]  
This represents a lower bound for convective heat transfer, given that there is no gas flow. We expect the heat transfer coefficient to be larger.
The experiment

The thermal conductivity of air is 0.014 Btu/ft-°F. If the sphere was 1 cm. in diameter, then

\[ h = 0.014 \left( \frac{1}{2.54 \times 12} \right) = 4.2 \text{ Btu/hr-ft}^2\text{-°F} \]

Note that \( h = 4.2 \text{ Btu/hr-ft}^2\text{-°F} = 0.00117 \text{ Btu/sec-ft}^2\text{-°F} \)

If \( \rho = 436 \text{ lb/ft}^3 \) and \( C_p = 0.12 \text{ Btu/lb.-°F} \), then

\[
\tau = \frac{0.00117 \times 6}{436 \times 0.12 \left( \frac{1}{30} \right)} = 0.00403 \text{ sec}
\]

If the heat transfer coefficient were 5 times larger (Nu = 10) then \( \tau = 0.02 \text{ sec} \),

If the heat transfer coefficient were 10 times larger (Nu = 100) then \( \tau = 0.2 \text{ sec} \),

It should be clear that we cannot make a reasonable verification of a uniform temperature until we solve for the temperature field in the sphere.