The Energy Balance

Consider a volume $\Omega$ enclosing a mass $M$ and bounded by a surface $\delta \Omega$.

At a point $x$, the density is $\rho$, the local velocity is $v$, and the local Energy density is $U$.

The rate of change total energy in $\Omega$ is:

$$\frac{d}{dt} \int \int \int _\Omega \rho U dV$$

The heat flow from the body is

$$\int \int _{\partial \Omega} q \cdot n \, dS$$

The Work done by the body on the surroundings is

$$\int \int _{\partial \Omega} v \cdot T \cdot n \, dS + \int \int \int _\Omega \rho g \cdot v dV$$

Since for the body

$$\dot{U} = \dot{Q} - \dot{W}$$

An equivalent form is

$$\frac{d}{dt} \int \int \int _\Omega \rho U dV + \int \int _{\partial \Omega} q \cdot n \, dS =$$

$$\int \int _{\partial \Omega} v \cdot T \cdot n \, dS + \int \int \int _\Omega \rho g \cdot v dV$$

If our control volume is a differential cube, the differential equation describing the Energy Equation is:

$$\frac{\partial}{\partial t} \rho U + \nabla \cdot (\rho U v) + \nabla \cdot q = \nabla \cdot (v \cdot T) + \rho g \cdot v$$
The first term is the local rate of energy change
The second is the convective energy flow
The third is the sum of reversible work and dissipation
The last is the work done by the gravitational acceleration.

Other Conservation Laws

Mass

\[ \frac{\partial}{\partial t} \rho + \nabla \cdot (\rho \mathbf{v}) = 0 \]

Momentum

\[ \frac{\partial}{\partial t} \rho \mathbf{v} + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) - \nabla \cdot \mathbf{T} - \rho \mathbf{g} = 0 \]

Mechanical Energy

This is obtained by taking the inner product of the momentum equation and the momentum equation to yield

\[ \mathbf{v} \cdot \left( \frac{\partial}{\partial t} \rho \mathbf{v} + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) - \nabla \cdot \mathbf{T} - \rho \mathbf{g} \right) = 0 \]

The real Energy Equation

The real Energy equation is obtained by subtracting the Mechanical Energy Balance from the complete Energy Equation, using the mass balance and recognizing that \( H = U + PV \).

\[ \rho \left( \frac{\partial}{\partial t} H + \mathbf{v} \cdot \nabla H \right) + \nabla \cdot \mathbf{q} = \tau \cdot \nabla \mathbf{v} - \dot{W}_s + \sum_{\alpha=1}^S \mathcal{R}_\alpha (\Delta H_\alpha) \]

This is simplified recalling that \( \left[ \frac{\partial H}{\partial T} \right]_p = C_p \)

\[ \rho C_p \left( \frac{\partial}{\partial t} T + \mathbf{v} \cdot \nabla T \right) + \nabla \cdot \mathbf{q} = \tau \cdot \nabla \mathbf{v} - \dot{W}_s + \sum_{\alpha=1}^S \mathcal{R}_\alpha (\Delta H_\alpha) \]
Applications of the Energy Equation to Steady State Conduction

The Energy Equation was

\[ \rho C_p \left( \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) + \nabla \cdot \mathbf{q} = \tau \cdot \nabla \mathbf{v} - \dot{W}_s + \sum_{\alpha=1}^{S} \mathcal{R}_\alpha \left( - \Delta H_\alpha \right) \]

But for systems at steady state where there is no motion, no shaft work done, and no chemical reaction

- time derivatives vanish
- the velocity, \( \mathbf{v} \), is zero
- the shaft work is zero, and
- the reaction rate is zero.

This means that the energy equation has a very simple form

\[ \nabla \cdot \mathbf{q} = 0 \]

Recall that Fourier's Law is a relation for the heat flux, \( \mathbf{q} \),

\[ \mathbf{q} = -k \nabla T \]

so that

\[ \nabla \left( k \nabla T \right) = 0 \]

and it follows for constant \( k \), that \( \nabla^2 T = 0 \)

In rectangular Cartesian coordinates, the resulting equation becomes the steady state heat conduction equation.

\[ \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \]
Boundary Conditions

Types of boundary conditions in heat transfer problems

1. Constant surface temperature

   On a surface S, the temperature is constant if
   \[ T(x, t) = T_s \]

2. Constant heat flux

   a) At a surface S, the flux is continuous, finite, and constant so that:
   \[ q_i = -k \left( \frac{\partial T}{\partial x_i} \right)_s \]

   b) At an adiabatic surface S, the flux vanishes:
   \[ q_i = -k \left( \frac{\partial T}{\partial x_i} \right)_s = 0 \]

3. Convective Surface condition

   At any surface, the flux leaving one body is equal to the flux leaving the other, so that
   \[ -k \left( \frac{\partial T'}{\partial x_i} \right)_s = -k \left( \frac{\partial T''}{\partial x_i} \right)_s \]
A Simple Steady State Conduction Problem

Consider a rectangular slab of infinite extent in the z-direction

_______ side is length L, the vertical sides are of length W.

The differential equation for steady heat conduction in 2 dimensions is:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

The boundary conditions are:

$$T = T_1 \text{ at } y = W$$
$$T = T_0 \text{ at } y = 0$$
$$T = T_0 \text{ at } x = 0$$
$$T = T_0 \text{ at } x = L$$
If the Temperature, $T$, and the independent variables, $x$ and $y$, are made dimensionless, as

$$
\Theta = \frac{T - T_0}{T_1 - T_0} \quad \text{and} \quad \eta = \frac{y}{W} \quad \text{and} \quad \zeta = \frac{x}{L}
$$

The conduction equation becomes

$$
\frac{\partial^2 \Theta}{\partial \zeta^2} + \left(\frac{L}{W}\right)^2 \frac{\partial^2 \Theta}{\partial \eta^2} = 0
$$

With the boundary conditions transformed to

$$
\Theta = 1 \text{ at } \eta = 1 \\
\Theta = 0 \text{ at } \zeta = 0 \\
\Theta = 0 \text{ at } \eta = 0 \\
\Theta = 0 \text{ at } \zeta = 1
$$

The method we use to solve this partial differential equation is "the method of separation of variables".

Assume that the solution is of the form

$$
\Theta = F(\zeta)G(\eta)
$$

We obtain an equation of the form

$$
\frac{\partial^2 F(\zeta)G(\eta)}{\partial \zeta^2} + \left(\frac{L}{W}\right)^2 \frac{\partial^2 F(\zeta)G(\eta)}{\partial \eta^2} = 0
$$
We group the terms that depend on each individual independent variable so that

\[ G(\eta) \frac{d^2 F(\zeta)}{d \zeta^2} + \left( \frac{L}{W} \right)^2 F(\zeta) \frac{d^2 G(\eta)}{d \eta^2} = 0 \]

If we divide by \( FG \) and separate variables, we obtain

\[ \frac{1}{F(\zeta)} \frac{d^2 F(\zeta)}{d \zeta^2} = - \left( \frac{L}{W} \right)^2 \frac{1}{G(\eta)} \frac{d^2 G(\eta)}{d \eta^2} = \text{constant} = - \lambda^2 \]

For simplicity, let \( \alpha^2 = \left( \frac{L}{W} \right) \)
The result is that we have two ordinary differential equations to solve:

so that

\[ \frac{d^2 F(\zeta)}{d\zeta^2} + \lambda^2 F(\zeta) = 0 \]

\[ \frac{d^2 G(\eta)}{d\eta^2} - \left(\frac{\lambda}{\alpha}\right)^2 G(\eta) = 0 \]

The solutions to the pair are:

\[ F(\zeta) = A \sin(\lambda \zeta) + B \cos(\lambda \zeta) \]
\[ G(\eta) = C \sinh\left(\frac{\lambda}{\alpha} \eta\right) + D \cosh\left(\frac{\lambda}{\alpha} \eta\right) \]

The entire solution is of the form

\[ \Theta = F(\zeta)G(\eta) = \left( A \sin(\lambda \zeta) + B \cos(\lambda \zeta) \right) \]
\[ \left( C \sinh\left(\frac{\lambda}{\alpha} \eta\right) + D \cosh\left(\frac{\lambda}{\alpha} \eta\right) \right) \]

If we recognize that since at $\zeta = 0$, $\Theta = 0$, then $B = 0$ and the solution simplifies considerably.

\[ \Theta = A' \sinh\left(\frac{\lambda}{\alpha} \eta\right) \sin(\lambda \zeta) + B' \cosh\left(\frac{\lambda}{\alpha} \eta\right) \sin(\lambda \zeta) \]
The boundary condition at $\eta = 0$ gives

$$\Theta = 0 \text{ at } \eta = 0 \text{ leads to } \quad 0 = A' \sinh(0) \sin(\lambda \zeta) + B' \cosh(0) \sin(\lambda \zeta)$$

So that $B'$ must be zero and the simplified solution is

$$\Theta = A' \sinh\left(\frac{\lambda}{\alpha} \eta\right) \sin(\lambda \zeta)$$

There are two constants left, $\lambda$ and $A'$, and two boundary conditions. The condition at $\zeta = 1$ leads to

$$0 = A' \sinh\left(\frac{\lambda}{\alpha} \alpha \right) \sin(\lambda)$$

And we must note that

Either $A'$ must vanish and the solution is trivial or

$$0 = \sin(\lambda)$$

This is true if and only if $\lambda = n\pi$ where $n = 0, 1, 2, 3, \ldots$.

That means that there are a countable infinite number of solutions. To find the solution we need to add all the possible solutions and determine the coefficients (constants).

$$\Theta = \sum_{n=1}^{\infty} \left( a_n \sinh\left(\frac{n\pi}{\alpha} \eta\right) \sin(n\pi \zeta) \right)$$
The coefficients may be determined by the last boundary condition.

\[ \Theta = 1 \text{ at } \eta = 1 \]

\[ 1 = \sum_{n=1}^{\infty} \left( a_n \sinh \left( \frac{n\pi}{\alpha} \right) \sin(n\pi \zeta) \right) \]

To determine the coefficient \( a_n \), we have to recognize the orthogonality property of sin functions, that is,

\[ \int_0^1 \sin(n\pi \zeta) \sin(m\pi \zeta) d\zeta = \begin{cases} 0 & \text{for } m \neq n \\ \frac{\pi}{2} & \text{for } m = n \end{cases} \]

To determine the coefficients, we can use the orthogonality properties of the sine and cosine functions.

\[ \sum_{n=1}^{\infty} \left( a_n \sinh \left( \frac{n\pi}{\alpha} \right) \right) \int_0^1 \sin(n\pi \zeta) \sin(m\pi \zeta) d\zeta = \int_0^1 (1) \sin(m\pi \zeta) d\zeta \]
Remember that the first sine integral is non-zero if and only if \( n = m \). Now the equation for \( a_n \) is

\[
a_n \frac{\pi}{2} \sinh \left( \frac{n\pi}{\alpha} \right) = \int_0^1 \sin(n\pi \zeta) d\zeta = \frac{-\cos \left( n\pi \zeta \right) \bigg|_0^1}{n}
\]

or

\[
a_n = \frac{2}{\pi} \frac{1 - (1)^n}{n \sinh \left( \frac{n\pi}{\alpha} \right)}
\]

Finally the solution is

\[
\Theta = \sum_{n=1}^{\infty} \left( \frac{2 \left(1 - (1)^n\right)}{n \pi} \frac{\sinh \left( \frac{n\pi}{\alpha} \right)}{\sinh \left( \frac{n\pi}{\alpha} \right)} \right) \sin \left( n\pi \zeta \right)
\]