CA 770A/ECE 673 Test 2

Two Hours; Closed Book/Notes
Calculators Allowed

Be sure to write your name on each page and show all work clearly.

Check to see that this test has seven questions in all.

Name: ______________________________

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Name: ____________________

(1) [5 marks] Suppose random variable X has probability distribution function $F(\cdot)$. Define another random variable, $Y = F(X)$. Show that Y is uniformly distributed over $[0, 1]$.

Name:________________________

(2) [10 marks] Consider an M/G/1 server with vacations. The server leaves on a vacation anytime the queue is empty. (If it comes back from a vacation to find the queue empty, it goes away on another vacation). The duration of the vacation is fixed and equal to 0.1 second. The arrival rate of jobs is λ jobs per second and the mean service time of a job is x seconds. Assume that $\lambda x < 1$. What is the average number of vacations taken by the server per second?

Name:________________________

(3) [20 marks] Draw the Markov chain associated with an M/M/2 queue with arrival rate λ and service rate μ per server. Write out its balance equations and, assuming that $\lambda < 2\mu$, obtain expressions for the steady-state probability of n jobs in the system.

Name:________________________

(4) [20 marks] Suppose you want to use importance sampling in simulating a coin toss. The coin has a probability $p_H = 0.1$ of coming up heads. You want to simulate to find the probability that, in a sequence of 3 tosses, you have a total of at least 2 heads. (In real life, you could simply write out an analytical expression for this probability, but that is beside the point here).

You use importance sampling by simulating a coin whose probability of coming up heads is 0.8. You run a total of 100 simulations (each simulation is for a run of 3 tosses) with this 0.8 probability. Each simulation can give you zero, one, two, or three heads. Of the 100 simulations, 30 give you 3 heads; 40 give you 2 heads; 20 give you 1 head; and the rest give you 0 heads.

Based on these statistics, what is your estimate of the probability of getting at least 2 heads for a coin with $p_H = 0.1$?

Name:____________________

(5) [20 marks] Consider a discrete-time Markov chain consisting of n states, numbered 1 to n. Let $q(i, j)$ be the transition probability of going from state i to state j. Show that if the chain is reversible, then the following condition must be satisfied:

For any finite sequence of states $j_1, j_2, \cdots, j_m$,

$$q(j_1, j_2)q(j_2, j_3)\cdots q(j_{n-1}, j_n)q(j_n, j_1) = q(j_1, j_n)q(j_n, j_{n-1})\cdots q(j_3, j_2)q(j_2, j_1)$$

Name:________________________

(6) [20 marks] In class, we used the disk replacement problem as an example of the use of Markov decision models. In that example, we assumed that the disk failure rate varied with the age of the disk.

By contrast, consider the simpler case where you have a disk which fails according to a *homogeneous* Poisson process, i.e., one with a *constant* failure rate, λ. The discount factor is α, the cost of replacing a functioning disk is r, and the cost of dealing with a failed disk is R. The rule is that when the disk fails, you are forced to replace it. What is the optimal policy with respect to the other states? That is, if the disk is functional and of age a, should you replace it (for $a = T, 2T, 3T, \cdots$ where you have the opportunity to replace only every T seconds)?

(Note: This problem should not require any calculations: just recognise the implications of having a fixed failure rate that is not a function of the disk age. However, you should be fairly rigorous in your answer.)

Name:____________________

(7) [10 marks] You are given a generator of random numbers, $U_1, U_2, \cdots,$, uniformly distributed over $[0, 1]$. Show how you would generate a random variable with the distribution, $F(x) = 1 - e^{-x^b}$ (for $x > 0$) where $b > 0$ is a given constant.

Name: ____________________

Blank Sheet: Use if you require additional space.