

Solutions to 2006 ECE 673 Final Examination

(1a) There are never more than $K + 1$ jobs in the system. There is a positive probability that there will be no arrivals during the service time of these jobs: this will idle the system. The key is that the queue size can never become infinite.

(1b) The throughput is $(1 - p_0)\mu$. Your gain is therefore

$$G = (1 - p_0)\mu y.$$

Your cost is

$$C = Kx.$$

The long-term profit per second is

$$G - C.$$

(2) Let $V(J_i)$ be the value of job i . $E[V(J_i)] = 0.8 \times 150 = 120$, for $i = 1, 2, 3$. We will accept job J_1 if its value is greater than or equal to the expected outcome from the next two interviews.

We will accept J_2 (if it is offered to us) only if $V(J_2) \geq E[V(J_3)]$, i.e., only if $V(J_2) \geq 120$. If we reject J_2 (or are rejected in this interview), we will have an expected gain of $E[V(J_3)] = 120$. So, we can condition on $V(J_2)$, and write the expected outcome from the last two interviews as follows:

- *Case 1:* $0 \leq V(J_2) < 120$: The probability of this is $0.2 + 0.2 \times 0.8 = 0.36$. The expected gain in this case would be $E[V(J_2)] = 120$.
- *Case 2:* $V(J_2) \geq 120$: The probability of this is $0.8 \times 0.8 = 0.64$. It is easy to show, using Bayes's law, that the density function of $V(J_2)$ under the condition that $V(J_2) \geq 120$ is $f_{v_2 > 120}(v) = 1/80$ for $120 \leq v \leq 200$.

The expected gain in this case is

$$E[V(J_2)|V(J_2) \geq 120] = 160.$$

Applying Bayes's law, we get the expected gain from the last two interviews as

$$G_{2,3} = 0.36 \times 120 + 0.64 \times 160 = 145.60.$$

We should therefore not accept the first job offer of 140.

(3) Suppose we want to generate a random variable X whose density function is

$$f(x) = x^2 e^{-x}/2, \quad x > 0,$$

by using the rejection method. The input to the rejection method is a stream of random variables with an exponential density having rate λ . Find the value of λ that minimizes the expected number of iterations of the algorithm used to generate X .

$g(x) = \lambda e^{-\lambda x}$. Find the value of x that maximizes ratio $f(x)/g(x)$ for a fixed λ . $\frac{f(x)}{g(x)} = \frac{1}{2\lambda} x^2 e^{(\lambda-1)x}$. If $\lambda \geq 1$, this will be maximized at $x = \infty$, and the maximal value will be ∞ . We therefore need to pick $\lambda < 1$. For $\lambda < 1$, we will maximize it at an x such that

$$\frac{d}{dx} x^2 e^{(\lambda-1)x} = 0 \Rightarrow x = \frac{2}{1-\lambda}.$$

For $x = 2/(1-\lambda)$, $\frac{f(x)}{g(x)} = \frac{2e^{-2}}{\lambda(1-\lambda)^2}$. We want to find λ to minimize this quantity: from calculus, we see that this value should be $\lambda = \frac{1}{3}$.

(4a) Let the average system time be $w + x$. Then, w is the expected time spent waiting for the other (later-arriving) jobs to be done. Condition on the number of busy periods, n , that are contained in the execution of one job. Then, we can write:

$$\begin{aligned} w &= \sum_{n=0}^{\infty} e^{-\lambda x} \frac{(\lambda x)^n}{n!} n(w + x) \\ &= \lambda x(w + x) \\ \Rightarrow w &= \frac{\lambda x^2}{1 - \lambda x} \end{aligned}$$

The system time is $w + x$. This is also the mean system time of the FIFO service discipline (can you see why?)

(4b) This is the number of jobs that would arrive over an interval of duration x , namely λx .

(4c) An attempt to complete the job will be successful if it is not preempted; this happens with probability $p = e^{-\lambda x}$. Define $q = 1 - p$. The expected number of times it will be preempted before it is done is then given by

$$\sum_{n=0}^{\infty} n q^n p = p (q + q^2 + q^3 + \dots) (1 + q + q^2 + \dots) = \frac{pq}{p^2} = \frac{q}{p}.$$

(5)

$$\begin{aligned}\text{Var}(X|Y) &= E[X^2|Y] - (E[X|Y])^2 \\ \Rightarrow E[\text{Var}(X|Y)] &= E[E[X^2|Y]] - E[(E[X|Y])^2] = E[X^2] - E[(E[X|Y])^2] \\ \text{Var}(E[X|Y]) &= E[(E[X|Y])^2] - (E[E[X|Y]])^2 \\ &= E[(E[X|Y])^2] - (E[X])^2 \\ \text{Var}(X) &= E[X^2] - E[(E[X|Y])^2] + E[(E[X|Y])^2] - (E[X])^2 \\ &= E[X^2] - (E[X])^2\end{aligned}$$