Solutions to ECE 673 Test 2

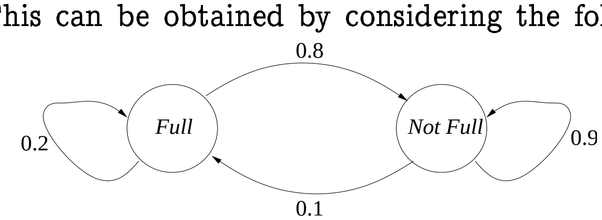
(1a) Let $r(t)$ be the residual time to the next bus arrival, $A(t)$ the number of arrivals over $[0,t]$, and X_i the interarrival time between the $i-1$'st and ith buses. The average residual time is

$$\begin{aligned}
\overline{R} &= \lim_{t\to\infty} \frac{1}{t}\int_0^\infty r(\tau)d\tau \\
&= \lim_{t\to\infty} \frac{A(t)}{t}\frac{\int_0^\infty r(\tau)d\tau}{A(t)} \\
&= \lim_{t\to\infty} \frac{A(t)}{t}\frac{\sum_{i=1}^n X_i^2}{2A(t)} \\
&= \frac{E[X^2]}{2E[X]} \\
E[X^2] &= 0.1\int_{10}^{20} x^2 dx = 700/3 \\
E[X] &= 15 \\
\overline{R} &= \frac{700}{90} = 7.78
\end{aligned}$$

(1b) Let N be the number of buses that are full. Then, the expected waiting time is

$$E[W] = \overline{R} + 15E[N].$$

To find N, we have to find the probability, p_1, that the first bus is full. Since we arrive at a random point in time, this probability is the steady-state probability that a bus is full. This can be obtained by considering the following Markov chain:



We have

$$\begin{aligned}
0.8p_f &= 0.1p_e \\
p_f &= 1 - p_e
\end{aligned}$$

where p_f, p_e are the probability of the bus being full and empty, respectively. Solving these equations yields $p_f = 1/9; p_e = 8/9$. Hence, the probability that we will have to miss N buses is given by:

$$p_m(N) = \begin{cases} p_e & \text{if } N = 0 \\ (0.2)^{N-1}0.8p_f & \text{otherwise} \end{cases}$$

We then have

$$E[N] = 0.8p_f \sum_{N=1}^{\infty} N(0.2)^{N-1} = 0.8p_f \frac{1}{(0.8)^2} = 0.138.$$

The expected waiting time is therefore given by

$$7.78 + 0.138 \times 15 = 9.85.$$

(2) $B^*(s) = e^{-10s}$. Denote the mean busy period by g. Hence, we have

$$\begin{aligned} G^*(s) &= e^{-10(s+\lambda-\lambda G^*(s))} \\ \Rightarrow \frac{dG^*(s)}{ds} &= -10\left(1 - \lambda\frac{dG^*(s)}{ds}\right) e^{-10(s+\lambda-\lambda G^*(s))} \\ \Rightarrow \left.\frac{dG^*(s)}{ds}\right|_{s=0} &= -10\left(1 - \lambda\frac{dG^*(s)}{ds}\right) \left.e^{-10(s+\lambda-\lambda G^*(s))}\right|_{s=0} \\ \Rightarrow -g &= -10(1 + \lambda g) \\ \Rightarrow g &= \frac{10}{1 - 10\lambda} \end{aligned}$$

(3) There are only two possible states here: $(0, 1), (1, 0)$, where (i, j) means there are i in the first system and j in the second. The balance equation is $\mu_1 p_{1,0} = \mu_2 p_{0,1}$. Combining this with the fact that the probabilities must add to one, we get the utilizations, u_1, u_2:

$$\begin{aligned} u_1 = p_{1,0} &= \frac{\mu_2}{\mu_1 + \mu_2} \\ u_2 = p_{0,1} &= \frac{\mu_1}{\mu_1 + \mu_2} \end{aligned}$$

(4) The service time is the sum of two exponentials. We can consider the service to have two stages, A and B, in which the time spent is exponentially distributed with parameters μ_1 and μ_2, respectively. The Markov chain is shown below and the balance equations can be written by inspection of it.