

CA 770A/ECE 673: Solutions to Test 2

(1) For $y \leq 0$, we clearly have $P(Y \leq y) = 0$ and for $y \geq 1$, $P(Y \leq y) = 1$. So, let us focus on $0 < y < 1$.

$$\begin{aligned} P(Y \leq y) &= P(F(X) \leq y) \\ &= P(X \leq F^{-1}(y)) \\ &= F(F^{-1}(y)) \\ &= y \end{aligned}$$

(2) If n is the number of vacations per second, the average time spent on vacation per second is $0.1n$. The server is either on vacation or serving jobs. Hence, its utilization is $1 - 0.1n$. But this must be equal to λx . This gives us

$$1 - 0.1n = \lambda x \Rightarrow n = 10(1 - \lambda x).$$

(3) The balance equations are:

$$\begin{aligned} \lambda p_0 &= \mu p_1 \\ (\lambda + \mu)p_1 &= \lambda p_0 + 2\mu p_2 \\ (\lambda + 2\mu)p_i &= \lambda p_{i-1} + 2\mu p_{i+1}, \quad i > 1 \end{aligned}$$

Define $\rho = \lambda/2\mu$.

From the balance equations, we have:

$$\begin{aligned} p_1 &= 2\rho p_0 \\ p_2 &= 2\rho^2 p_0 \\ p_3 &= 2\rho^3 p_0 \\ &\vdots \\ p_i &= 2\rho^i p_0 \end{aligned}$$

Using the fact that the probabilities must sum to 1, we get:

$$\begin{aligned} (1 + 2\rho + 2\rho^2 + \dots) p_0 &= 1 \\ \Rightarrow p_0 &= \frac{1 - \rho}{1 + \rho} \end{aligned}$$

(4) The sample path associated with each simulation has to be “reweighted” to correct for the fact that we simulated with a p_H of 0.8 rather than 0.1. The probability that we get 3 heads is greater in the simulation by a factor of $(0.8/0.1)^3 = 256$; and the probability that we get one particular sequence of 2 heads and 1 tail is greater by the factor $(0.8/0.1)^2(0.2/0.9) = 14.2222$. Hence, our estimate of the required probability is

$$\frac{30/256 + 40/14.2222}{100} = 0.029.$$

(5) In class, we defined a reversible discrete-time Markov chain as one for which

$$\pi(i)q(i, j) = \pi(j)q(j, i)$$

for every two states i, j in the chain. From this, we can write:

$$\begin{aligned} \pi(j_1)q(j_1, j_2) &= \pi(j_2)q(j_2, j_1) \\ \pi(j_2)q(j_2, j_3) &= \pi(j_3)q(j_3, j_2) \\ &\vdots \\ \pi(j_n)q(j_n, j_1) &= \pi(j_1)q(j_1, j_n) \end{aligned}$$

Multiplying the LHS and RHS of these equations, we achieve the desired result.

(6) The only time you will replace the disk is when it fails. The reason is that the disk failure process is independent of the age of the disk. Hence, replacing a functioning disk is not going to have any impact on whether a failure is going to occur in the future. That is, the probabilities $P_{i,j}(a)$ are independent of the action a .

Now, consider the expression for the reward model:

$$V_h(i) = \min_a \left\{ C(i, a) + \alpha \sum_j P_{i,j}(a) V_h(j) \right\}.$$

Since $P_{i,j}(a)$ is independent of the action a , we can write:

$$V_h(i) = \min_a \{ C(i, a) \} + \alpha \sum_j P_{i,j}(a) V_h(j).$$

From this, it is easy to see that *any* policy that replaces a functional disk is more expensive than one which replaces only when the disk fails.

(7) Use the inverse-transform method:

$$\begin{aligned}U &= 1 - e^{-x^b} \\ \Rightarrow 1 - U &= e^{-x^b} \\ \Rightarrow \ln(1 - U) &= -x^b \\ \Rightarrow x &= \sqrt[b]{1 - U}\end{aligned}$$

For obvious reasons, we can also use $x = \sqrt[b]{U}$.