CA 770A/ECE 673: Solutions to Test 1

(1) We leave behind an empty queue, so the time to the next departure is the time to the next arrival plus the service time of that arrival. Using this fact, we can write:

$$\begin{aligned} f_{inter-departure}(t) &= \int_{\tau=0}^{t} \lambda e^{-\lambda t} \times \frac{\lambda}{2} e^{-\lambda(t-\tau)/2} \\ &= \lambda e^{-\lambda t/2}\left(1 - e^{-\lambda t/2}\right) \end{aligned}$$

(2) Condition on the vacation time: if ν is the vacation time then the probability that the server comes back to find the system still empty is given by $e^{-\lambda\nu}$. Now, use the fact that ν has pdf $\alpha e^{-\lambda\alpha}$ and uncondition on ν, using Bayes's Law. We get the required probability as:

$$\begin{aligned} \int_{\nu=0}^{\infty} e^{-\lambda\nu} \alpha e^{-\lambda\alpha} d\nu &= \alpha \int_{\nu=0}^{\infty} e^{-(\lambda+\alpha)\nu} d\nu \\ &= \frac{\alpha}{\alpha+\lambda} \end{aligned}$$

This is something we can see intuitively from the definitions of α and λ.

(3) (a) Use the inverse-transform method. Calculate $P\{X = i\}$ for $i = 0, \cdots, k$. Compute $\Pi_\ell = \sum_{i=0}^{\ell} P\{X = i\}$. Generate U_i, uniformly distributed between 0 and 1. Find i such that $\Pi_{i-1} \leq U_i < Pi_i$ and output i. (See page 45 of the book for this).

(b) This is the sum of two random variables, which we can generate separately and then add. Generate $Y_1 = -\frac{1}{\mu} \ln U_1$, and set $Y_2 = U_2$. Output $Y_1 + Y_2$.

(4) Find the sample mean: it is given by

$$\overline{X} = \frac{100 \times 10 + 150 \times 12 + 20 \times 250}{500} = 15.6.$$

The sample variance is computed using the expression

$$S = \sum_{i=1}^{n} \frac{(X_i - \overline{X})^2}{n-1} = 19.88.$$

To find the confidence intervals, turn to the table of the area under the standard normal curve. The numbers in this table show the area from $-\infty$ to x; however, we want to pick y such that the area from $-y$ to y is 0.8 for the 80% confidence interval and 0.95 for the 95% confidence interval. We exploit the fact that the standard normal density is symmetric about $y = 0$, which means that we must use x from the table when the area from $-\infty$ to x is 0.9 and 0.975 respectively, for the 80% and 95% confidence cutoffs. Looking at the table provided with the test, we see that $x = 1.28$ gives an area of 0.8997 and $x = 1.29$ an area of 0.9015. Interpolating, we pick $x = 1.2817$ for the 80% confidence interval. The confidence interval is therefore

$$\left[\overline{X} - 1.2817 \times 19.88/\sqrt{500}, \overline{X} + 1.2817 \times 19.88/\sqrt{500}\right] = [14.46, 16.74].$$

For the 95% confidence interval, we pick $x = 1.96$ (i.e., 0.95 is the area under the standard normal curve from $[-1.96, 1.96]$.) This gives us a confidence interval of $[13.86, 17.34]$.

For the 100% confidence interval, we note from our knowledge of the normal density that the area under the normal curve goes to 1 only for $x = \infty$. The 100% confidence interval is therefore $[-\infty, \infty]$.

(5) Let $f(p)$ be the density function representing our knowledge of p_H. We are asked to compute $f(p|100 \text{ heads out of 100 tosses})$. To do this, we use Bayes's law:

$$\begin{aligned}
f(p|100 \text{ heads out of 100 tosses}) &= \frac{P(100 \text{ heads out of 100 tosses}|p_H = p)f(p)}{P(100 \text{ heads out of 100 tosses})} \\
&= \frac{p^100}{P(100 \text{ heads out of 100 tosses})} \\
P(100 \text{ heads out of 100 tosses}) &= \int_{p_H=0}^{1} P(100 \text{ heads out of 100 tosses}|p_H = p)f(p)\,dp \\
&= \int_{p_H=0}^{1} p^{100}\,dp \\
&= 1/101 \\
f(p|100 \text{ heads out of 100 tosses}) &= 101p^{100}
\end{aligned}$$