ECE 673: Solutions to Homework 5

(2) The probability that output line j is busy is

$$\pi_j = 1 - \prod_{i=0}^{n-1}(1 - p_{i,j}).$$

The expected bandwidth is the sum of the probabilities of the lines being busy:

$$E[BW] = \sum_{j=0}^{m-1} \pi_j.$$

(3) Similar to an M/M/1 Markov chain, except that the rate from state i to state $i-1$ is $i\mu$. The differential equations are:

$$\begin{aligned}
\frac{d\pi_0(t)}{dt} &= -\lambda\pi_0(t) + \mu\pi_1(t) \\
\frac{d\pi_i(t)}{dt} &= -(\lambda + i\mu)\pi_i(t) + \lambda\pi_{i-1}(t) + (i+1)\mu\pi_{i+1}(t), \quad i > 0
\end{aligned}$$

(4) A busy period begins upon an arrival to an empty queue and lasts until the next visit of that queue to state 0. So, we can simply alter the $M/M/\infty$ Markov chain by removing all transitions out of state 0, i.e., by making state 0 an absorbing state. Write the differential equations for such a chain. Solve the differential equations with the initial condition that $\pi_1(0) = 1$. Then, the probability distribution function of the busy period is given by $G(t) = \pi_0(t)$. (Can you see why this works?)

(5a) The crosswalk is being modeled as an $M/M/\infty$ queue. One way of solving for the probability of coming upon an empty queue is to write out the balance equations for the queue and then solving for π_0. We can solve the equations either directly or by using z-transforms.

Solving the equations directly yields $p_i = (\rho^i/i!)p_0$ where $\rho = \lambda/\mu$. Using the condition that the probabilities must sum to one, we end up with $p_0 = e^{-\rho}$.

Using z-transforms is a bit more complicated. As usual, multiply the i'th balance equation by z^i on both sides and then add the equations. We will get:

$$\begin{aligned}
\lambda p_0 &= \mu p_1 \\
(\lambda + \mu)p_1 z &= \lambda p_0 z + 2\mu p_2 z \\
(\lambda + 2\mu)p_2 z^2 &= \lambda p_1 z^2 + 3\mu p_3 z^2 \\
&\vdots \\
(\lambda + i\mu)p_i z^i &= \lambda p_{i-1} z^{i-1} + (i+1)\mu p_{i+1} z^i
\end{aligned}$$

After some minor manipulation, we obtain the following differential equation:

$$
\begin{aligned}
\lambda(1-z)P(z) &= \mu(1-z)\frac{dP(z)}{dz} \\
\Rightarrow \frac{dP(z)}{P(z)} &= \rho dz \\
\Rightarrow \ln P(z) &= \rho z + K, \;\; K \text{ is a constant of integration} \\
\Rightarrow P(z) &= e^{\rho z + K} \\
P(1) = 1 &\Rightarrow K = -\rho \Rightarrow P(z) = e^{-\rho(1-z)} \\
\Rightarrow p_0 &= P(0) = e^{-\rho}
\end{aligned}
$$

There is a much simpler way to calculate p_0. The average time between the end of one busy period and the beginning of another is $t_{idle} = 1/\lambda$. The average duration of a busy period is given to be θ. So, we have alternating busy and idle periods, with mean times t_{idle} and θ, respectively. Hence the probability that we will arrive at a random instant and find the crosswalk empty of pedestrians is

$$p_0 = \frac{t_{idle}}{t_{idle} + \theta}.$$

(A byproduct of this calculation is that since we also know that $p_0 = e^{-\rho}$, we now have an expression for the average busy period, θ, in this queue.)

The probability that we do not need to stop at all is given by p_0^5.

(5b) We have to calculate the average residual time of the busy period. We already know how to calculate average residual times: see the discussion of the vacation model. The average residual time is given by $\phi/(2\theta)$ (in the vacation model, we showed it was $\overline{V^2}/(2\overline{V})$). This is the delay if we have to stop; if not, the waiting time is 0. Hence, the total expected time is

$$5(1-p_0)\frac{\phi}{2\theta}.$$