

ECE 673: Solutions to Homework 4

(5.13a) The explanation for this is similar to its analogue for the M/G/1 queue.

(5.13b) Let $f(t)$ be the density function of the interval between successive instants. Then, the probability mass function of the number of jobs that arrive in this interval can be written out using Bayes's law:

$$\begin{aligned} P(v = n) &= \int_{t=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} f(t) dt \\ \sum_{n=0}^{\infty} P(v = n) z^n &= \sum_{n=0}^{\infty} \int_{t=0}^{\infty} e^{-\lambda t} \frac{(\lambda t z)^n}{n!} f(t) dt \\ &= \int_{t=0}^{\infty} e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t z)^n}{n!} f(t) dt \\ &= \int_{t=0}^{\infty} e^{-(\lambda - \lambda z)t} f(t) dt \\ &= F^*(\lambda - \lambda z) \end{aligned}$$

One convenient way to represent the underlying recursion is to use the indicator variable, 1_X , which is defined as being 1 if X is true and 0 otherwise. Defining $X = (q_n \geq m)$, we can therefore write

$$q_{n+1} = (q_n - m)1_X + v_n.$$

From this point onwards, the derivation follows the same pattern as the standard M/G/1 deriva-

tion.

$$\begin{aligned}
E[z^{q_{n+1}}] &= E[z^{(q_n - m)1_X}]E[z^{v_n}] \\
E[z^q] &= E[z^{(q-m)1_X}]E[z^v] \quad \text{in the limit as } n \rightarrow \infty \\
E[z^{(q-m)1_X}] &= z^0 P(q_n < m) + p_m + p_{m+1}z + p_{m+2}z^2 + \dots \\
&= \sum_{i=0}^{m-1} p_i + \sum_{i=m}^{\infty} p_i z^{i-m} \\
&= \sum_{i=0}^{m-1} p_i + z^{-m} \sum_{i=m}^{\infty} p_i z^i \\
&= \sum_{i=0}^{m-1} p_i + z^{-m} \left(\sum_{i=0}^{\infty} p_i z^i - \sum_{i=0}^{m-1} p_i z^i \right) \\
&= z^{-m} Q(z) + \sum_{i=0}^{m-1} p_i (1 - z^{i-m}) \\
\Rightarrow Q(z) &= \left(z^{-m} Q(z) + \sum_{i=0}^{m-1} p_i (1 - z^{i-m}) \right) F^*(\lambda - \lambda z) \\
\Rightarrow z^m Q(z) [F^*(\lambda - \lambda z)]^{-1} &= Q(z) + \sum_{i=0}^{m-1} p_i (z^m - z^i) \\
\Rightarrow Q(z) &= \frac{\sum_{i=0}^{m-1} p_i (z^m - z^i)}{z^m [F^*(\lambda - \lambda z)]^{-1} - 1}
\end{aligned}$$

(5.19) If $w(t)$ is the waiting time density, then the expected cost is

$$\int_{t=0}^{\infty} w(t) \alpha e^{\beta t} dt = \alpha W^*(-\beta) = \frac{\alpha \beta (1 - \rho)}{\beta + \lambda - \lambda B^*(-\beta)}.$$

For the cost to be finite, we clearly need

$$\beta + \lambda - \lambda B^*(-\beta) > 0.$$

(5.20a)

$$\begin{aligned}
D^*(s) &= D^*(s|\text{empty})P(\text{empty}) + D^*(s|\text{nonempty})P(\text{nonempty}) \\
&= \frac{\lambda}{s + \lambda} B^*(s)(1 - \rho) + \rho B^*(s) \\
d(t) &= a \oplus b(t)(1 - \rho) + \rho b(t)
\end{aligned}$$

where \oplus is the convolution operator. (Recall that the inverse of the product of two Laplace transforms is the convolution of their respective inverses).

(5.20b) $b(t)$ is a delta function at time T . Denote this by δ_T . Then, the interdeparture time density is

$$d(t) = \begin{cases} \rho & \text{if } t = T \\ (1 - \rho)a(t - T) & \text{otherwise} \end{cases}$$

$$(5.22a) \quad B_T^*(s) = \sum_{i=1}^{\infty} p^{i-1} q (B^*(s))^i = q B^*(s) / (1 - p B^*(s)).$$

(5.22b)

$$\begin{aligned} E[x_T] &= -dB_T^*(s)/ds|_{s \rightarrow 0} \\ &= \bar{x}/q \text{ after some calculus} \\ E[x_T^2] &= d^2B_T^*(s)/ds^2|_{s \rightarrow 0} \\ &= \overline{x^2}q^{-2} + 2p\bar{x}^2q^{-2} \end{aligned}$$

(5) Since the arrival rate at each queue is the same and so is the service time of each packet, and service is exhaustive in each queue, it is clear that $Q_i(z) = Q_j(z)$. So, we simply drop the subscript and call it $Q(z)$.

When the token arrives, it finds i jobs in the queue with probability q_i . Since it does not leave until the queue is empty, the number of jobs it serves is the number of jobs in i independent busy periods. The transform of this is $(G^*(s))^i$. So, the transform of the number of jobs served in a token visit is given by

$$N(z) = \sum_{i=0}^{\infty} q_i (G^*(s))^i = Q(G^*(s)) \quad (1)$$

The transform of the total number of jobs served between leaving queue i and returning to it is then approximately equal to $M(z) = (N(z))^{n-1}$. The approximation we are making is that the number of jobs served in each node is independent of the number served in any other node (which is obviously not true). If m_i is the probability that this total number is i , then the time taken to serve these jobs has the transform $T^*(s) = \sum_{i=0}^{\infty} m_i (B^*(s))^i = M(B^*(s))$. (We are given that $B^*(s) = e^{-st_p}$). To the packet service time, we must add the time taken to pass the token a total of n times: the transform of this time is e^{-snt_p} . The total time that elapses between when the token departs queue i and when it next returns to it is therefore given by $\Theta^*(s) = e^{-snt_p} T^*(s)$. The generating function of the number of jobs that arrive (in queue i) during this residual cycle time is given by $\Theta^*(\lambda - \lambda z)$. Hence, we have the implicit equation:

$$Q(z) = \Theta^*(\lambda - \lambda z).$$

(It is implicit because Θ is itself a function of $Q(z)$).

(6) First, write out the equations that the traffic rates must satisfy: In matrix form, these would be:

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_6 \end{pmatrix} = R \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_6 \end{pmatrix}.$$

These are dependent equations, so we simply set $\lambda_1 = 1$ and calculate the others with respect to this one. To do so, since we have six equations in five unknowns, we can simply drop one of these equations. It doesn't matter which we drop, since these 6 equations are dependent and so there is redundancy that preserves the conditions of the dropped equation.

Then, define $\rho_i = \lambda_i/\mu_i$. These are relative utilizations which need to be normalized. Now, use Buzen's algorithm to calculate $G(N, M)$. The probability $P_i(n_i) = \rho_i^{n_i} \frac{G(N-i, M)}{G(N, M)}$.