ECE 673: Solutions to Homework 3

(4.9) The transitions associated with arrivals take the state from i to $i+1$ for $i = 0, 1, \cdots$. The departure transitions from state 1 go to so state 0 and those from state i to $i-2$, for $i = 2, 3, \cdots$. The arrival rate is λ and the departure rate is μ. The balance equations are:

$$\begin{aligned} \lambda\pi_0 &= \mu(\pi_1 + \pi_2) \\ (\lambda + \mu)\pi_i &= \lambda\pi_{i-1} + \mu\pi_{i+2}, i = 1, 2, \cdots \end{aligned}$$

Multiplying the i'th equation by z^i and summing, we get:

$$\begin{aligned} \sum_{i=0}^{\infty} \lambda\pi_i z^i + \sum_{i=1}^{\infty} \mu\pi_i z^i &= \mu\pi_1 + \mu\pi_2 + \sum_{i=1}^{\infty} \lambda\pi_{i-1} z^i + \sum_{i=1}^{\infty} \mu\pi_{i+2} z^i \\ \Rightarrow \lambda\Pi(z) + \mu(\Pi(z) - \pi_0) &= \mu\pi_1 + \mu\pi_2 + \lambda z\Pi(z) + \mu z^{-2}(\Pi(z) - \pi_0 - \pi_1 z - \pi_2 z^2) \\ \Rightarrow \Pi(z) &= \frac{\mu(z^2 - 1)\pi_0 + \mu(z^2 - z)\pi_1}{-\lambda z^3 + (\lambda + \mu)z^2 - \mu} \\ &= \frac{(z+1)\mu\pi_0 + z\mu\pi_1}{-\lambda z^2 + \mu z + \mu} \end{aligned}$$

Now, use the fact that $\lim_{z\to 1} \Pi(z) = 1$, i.e.,

$$\begin{aligned} \frac{2\mu\pi_0 + \mu\pi_1}{-\lambda + 2\mu} &= 1 \\ \Rightarrow \pi_1 &= -\lambda/\mu + 2 - 2\pi_0 \end{aligned}$$

(5.4) Set $K = 1 - \rho$. Then,

$$\begin{aligned} Q(z) &= V(z)\frac{K(1 - 1/z)}{1 - V(z)/z} \\ &= \frac{KzV(z) - KV(z)}{z - V(z)} \end{aligned}$$

If we substitute $z = 1$ in the above expression, we get $0/0$. Hence, we use L'Hospital's rule for $\lim_{z\to 1} Q(z)$, and get

$$\begin{aligned} \frac{KV(1) + KV'(1) - KV'(1)}{1 - V'(1)} &= 1 \\ \Rightarrow K &= 1 - V'(1) = 1 - \lambda\overline{x} \\ \Rightarrow \rho &= \lambda\overline{x} \end{aligned}$$

[0]This homework involved a great many algebraic expressions. If you notice a typo anywhere in this document, please let me know.

(5.11) The probability that n groups arrive in $[0, t]$ is $e^{-\lambda t}(\lambda t)^n/n!$. The z-transform of the total number of jobs if n groups arrive is therefore $e^{-\lambda t}(\lambda t G(z))^n/n!$. Now, uncondition on n to get the unconditional transform of the total number of jobs:

$$\sum_{n=0}^{\infty} e^{-\lambda t}(\lambda t G(z))^n/n! = e^{-\lambda t(1-G(z))}.$$

(5.16) For convenience, define $q = 1 - p$.

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\begin{aligned}
E[X] &= E[X|\text{head}]P(\text{head}) + E[X|\text{tail}]P(\text{tail}) \\
&= 0 \cdot p + (1/p) \cdot q \\
&= q/p \\
E[X^2] &= E[X^2|\text{head}]P(\text{head}) + E[X^2|\text{tail}]P(\text{tail}) \\
&= 0 \cdot p + (2/p^2)q \\
\sigma_b^2 &= E[X^2] - (E[X])^2 \\
&= 2q/p^2 - q^2/p^2 \\
&= (1-p^2)/p^2 \\
W &= \overline{N}/\lambda \\
&= \frac{\lambda \overline{x^2}}{2(1-\rho)} \\
\rho &= \lambda q/p = \lambda/p - 1 \\
W &= \frac{\lambda 2q/p^2}{2(1 - q\lambda/p)} \\
&= \frac{q\lambda}{p(p - q\lambda)} \\
B^*(s) &= p + \frac{pq}{s+p} \\
W^*(s) &= \frac{s(1-\rho)}{s - \lambda + \lambda B^*(s)} \\
&= \frac{s(1 - \lambda q/p)}{s - \lambda + \lambda\left(p + \frac{qp}{s+p}\right)} \\
&= \frac{(1-\rho)(s+p)}{s + p - \lambda + \lambda p} \\
&= \frac{(2 - \lambda/p)(s+p)}{s + p - \lambda + \lambda p} \\
W^{*\prime}(s) &= \frac{(p - \lambda + \lambda p)(s + p - \lambda + \lambda p) - (p+s)(p - \lambda + \lambda p)}{p(s + p - \lambda + \lambda p)^2} \\
W &= -W^{*}(0) = \frac{\lambda(1-p)}{p(p - \lambda + \lambda p)} \\
W^*(s) &= \frac{(1-\rho)(s+p)}{s + p - \lambda + \lambda p} \\
&= 1 - \rho + \frac{p\rho - p\rho^2}{s + p - \lambda + \lambda p} \\
\Rightarrow w(t) &= \begin{cases} 1 - \rho & \text{if } t = 0 \\ (p\rho - p\rho^2)e^{-(p - \lambda + \lambda p)t} & \text{if } t > 0 \end{cases}
\end{aligned}
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