

ECE 673: Homework 2

Due: March 9 (On-Campus Students);
One week after watching Lecture 9 (Off-campus students).

(1) The balance equations are:

$$\begin{aligned}\lambda\pi_0 &= \mu\pi_1 \\ (\lambda + \mu)\pi_1 &= \mu\pi_2 \\ (\lambda + \mu)\pi_i &= \lambda\pi_{i-2} + \mu\pi_{i+1}, \quad i \geq 2\end{aligned}$$

Multiplying the i 'th equation by z^i and summing, we get:

$$\begin{aligned}(\lambda + \mu) \sum_{i=1}^{\infty} \pi_i z^i + \lambda\pi_0 &= \mu \sum_{i=1}^{\infty} i \pi_i z^{i-1} + \lambda \sum_{i=2}^{\infty} \pi_{i-2} z^i \\ \Rightarrow (\lambda + \mu)(\Pi(z) - \pi_0) + \lambda\pi_0 &= \mu z^{-1}(\Pi(z) - \pi_0) + \lambda z^2 \Pi(z) \\ \Rightarrow \Pi(z) &= \frac{\mu(z-1)}{-\lambda z^3 + (\lambda + \mu)z - \mu} \pi_0 \\ &= \frac{\mu\pi_0}{-\lambda z^2 - \lambda z + \mu}\end{aligned}$$

Since $\Pi(1) = 1$, we have $\pi_0 = 1 - \rho$, where $\rho = 2\lambda/\mu$.

(2) A Poisson process with rate λ has the following defining characteristics:

- (a) The probability of an arrival in an interval of duration t is $\lambda t + o(t^2)$.
- (b) Arrivals in disjoint intervals are independent of one another.
- (c) The probability of more than one arrival in an interval of duration t is $o(t^2)$.

Consider the union of two independent Poisson processes, of rates λ_1 and λ_2 , respectively.

The probability of no arrivals in an interval of duration t is

$$(1 - \lambda_1 t + o(t^2))(1 - \lambda_2 t + o(t^2)) = 1 - (\lambda_1 + \lambda_2)t + o(t^2).$$

This establishes (a).

It is trivial to see that if the component processes are Poisson, that arrivals in disjoint intervals must be independent. This establishes (b).

For (c), note that more than one arrival in the combined process may be due to multiple arrivals in either process, or one arrival each in the two processes. Over an interval t , the

probability of each of these quantities is $o(t^2)$.

(3) If the state is i , there are $i - 1$ jobs in the queue and one being served (for $i > 0$). A job in position j of the queue has a departure rate (due to impatience) of $j\gamma$. In addition, the job in service has a departure rate of μ . Hence, we have a birth-death process, in which the birth rate in state i is λ and the death rate is $\mu + (1 + \dots + i - 1)\gamma$ for $i > 1$. This leads to the balance equations:

$$\begin{aligned} \lambda\pi_0 &= \mu\pi_1 \\ \left(\lambda + \mu + \frac{i(i-1)}{2}\gamma\right)\pi_i &= \lambda\pi_{i-1} + \mu\pi_{i+1}. \end{aligned}$$

(4) Define the state to be the number of functional processors. Then, we have a birth-death process, in which state i has a birth rate of μ (corresponding to repair) for $i = 0, 1, 2, 3, 4$, and a death rate of $i\lambda$. The balance equations are:

$$\begin{aligned} \mu\pi_0 &= \lambda\pi_1 \\ (\mu + i\lambda)\pi_i &= i\lambda\pi_{i+1} + \mu\pi_{i-1}, \quad i = 1, 2, 3, 4. \end{aligned}$$

Solving these in terms of π_0 , we obtain $\pi_i = (\rho^i/i!)\pi_0$. Use the boundary condition that the probabilities must sum to 1 to solve for π_0 :

$$\pi_0 = \left(\sum_{i=0}^5 \frac{\rho^i}{i!}\right)^{-1}.$$