ECE 673: Homework 6

Due: May 2, 2006.

In questions 1 and 2, you will need a good random number generator. You can reuse the one you picked in Homework 1. For both questions, hand in your fully documented code.

(1) Consider using random numbers to evaluate the following integral:

$$A = \int_0^{10} e^{-(x^2+x^3)} dx.$$

As mentioned in class, we can obtain the numerical value of this integral by pretending that we have a random variable, $A(X) = e^{-(X^2+X^3)}$, where X is uniformly distributed over $[0, 10]$. The calculation of the integral reduces to estimating $E[A(X)]$.

- (a) Obtain 95% confidence intervals for this integral by generating $A(X_1), A(X_2), \cdots, A(X_m)$ for $m = 1000$.
- (b) Find the confidence intervals using the method of antithetic variables. Generate pairs of random variables, X_i' and X_i'', which are negatively correlated with respect to each other, define $Y_i = (X_i' + X_i'')/2$, and calculate the variance based on Y_i, $i = 1, \cdots, 500$.
- (c) Use the method of stratified sampling to generate a total of 1000 samples of this function. Divide the interval $[0, 10]$ into 20 segments, each of length 0.5 (this is NOT necessarily the best way to divide the interval). Then, pick 50 instances of X_i falling in each segment. Find the confidence interval for the estimate of A associated with this approach.

(2) Random graphs are finding an increasing number of applications. In this question, we will evaluate the connectedness of such graphs.

Consider a random directed graph of $N = 20$ nodes, whose edges are generated as follows. For each node, v, in the graph, generate its out-degree $d(v)$, uniformly distributed over the set $\{1, 2, 3, 4\}$. Then, pick at random $d(v)$ of the nodes and draw edges from node v to these randomly chosen nodes.

In your program, you can represent the graph using an adjacency matrix (look up any book on algorithms if you have forgotten the basics of graph algorithms). Use any algorithm you like to determine the probability that such a graph is connected, i.e., that it has a path from every node to every other node.

- (a) Use standard simulation. Generate 1000 instances of random graphs and find the probability that such graphs are connected. Find the 95% confidence interval associated with this probability.

- (b) Use the method of control variates to obtain this probability and its 95% confidence interval. One obvious control variate is the total number of edges of the graph. Again, simulate for 1000 graphs.

(3) [From Question 10 of Chapter 8 (Ross)] In certain situations, a random variable Y, whose mean is known, is simulated so as to obtain an estimate of $P\{Y \leq a\}$ for a given constant, a. The raw simulation estimator from a single run is I, where

$$I = \begin{cases} 1 & \text{if } Y \leq a \\ 0 & \text{otherwise} \end{cases}$$

Because I and Y are negatively correlated, we can use Y as a control variate, and use an estimator of the form $Z = I + c(Y - E[Y])$.

Assume that Y is uniformly distributed over $[0, 1]$. What is the best value of c to use? Determine theoretically (i.e., without simulation, but directly by analysis) the ratio

$$\frac{\mathrm{Var}(I)}{\mathrm{Var}(Z)}.$$