A Modern Extreme Value Theory Approach to Calculating the Distribution of the Peak-to-Average Power Ratio in OFDM Systems

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***Abstract*—Orthogonal frequency division multiplexing (OFDM) is a promising framework for future wireless communication systems. One of the main impediments that has limited the applicability of OFDM systems in low-power wireless communication systems is the highly variable amplitude of the baseband transmitted signal; thus, a number of recent analyses have characterized this variation. These analyses have generally employed the following two components: (1) the *assumption* that the complex envelope of the OFDM signal converges to a Gaussian random process in some sense as the number of subcarriers becomes large, and (2) Rice's classical results on level-crossing rates for the envelope of Gaussian random processes. In this work, we improve on both of these components to arrive at a simple, accurate, and rigorously-established expression for the peak distribution of the OFDM envelope. In particular, using a rigorous (and non-trivial) proof establishing the convergence in (1) above as justification, the modern extreme value theory for chi-squared processes is applied to the problem. Numerical results for both uncoded and coded systems establish that the simple expression obtained for the distribution of the peaks of the envelope process is extremely accurate, even for a modest number of subcarriers.**

I. Introduction

A major goal of modern communication systems is to allow high-speed communication, regardless of the location or mobility of the system users. However, this goal is difficult to achieve due to the multipath fading that affects wireless communication signals. One alternative for achieving high-speed wireless communication in the presence of multipath fading is to employ a multicarrier system, generally implemented as an orthogonal frequency division multiplexing (OFDM) system, in conjunction with error control coding. Such coded OFDM systems have emerged recently as a serious competitor to single-carrier systems and have been employed or are being considered for a number of applications, including digital audio broadcast and digital video broadcast in Europe, wireless local area networks, broadband fixed wireless access, and cellular data.

One of the challenges to be overcome when employing an OFDM system in low-power peer-to-peer wireless communication systems is that the complex envelope of the baseband OFDM signal can demonstrate significant variation; in other words, its peak-to-mean envelope power ratio (PMEPR) can be much larger than that of an analogous single-carrier system.

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This large PMEPR requires significant backoff of the average operating power in the amplifier in the transmitter if it is to be operated in the linear region, which results in significant power inefficiency. Thus, there has been a large body of work in the analysis of the variation of the complex envelope of the OFDM signal and in methods to reduce this variation. Here, the focus is on the analysis problem. Recent papers that have analyzed the PMEPR of the transmitted OFDM signal [1] [2] [3] often assume that the complex envelope of the transmitted OFDM signal converges in some sense to a Gaussian random process as the number of subcarriers becomes large, and then use this assumption to justify the use of the level-crossing results of Rice [4] for the envelope of a complex Gaussian random process to study the PMEPR distribution.

In this paper, we arrive at a simple, rigorously justified, and accurate expression for the distribution for the PMEPR distribution and show that it applies for both uncoded and coded OFDM systems. The rigorous justification relies on the *proof* that a bandlimited OFDM signal converges weakly to a certain Gaussian random process. The main theorem, for which the full proof is given in [5], is

Theorem 2

Consider the complex signal

$$s_N(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} A_k e^{j\omega_k t}, \tag{1}$$

where $\omega_k = \frac{2\pi k}{NT_c}$, $T_c \in (0,\infty)$, and $\{A_k,\, k=0,\cdots,N-1\}$ is an independent and identically distributed (IID) sequence of complex random variables, each with independent real (A_k^R) and imaginary (A_k^I) parts such that $E\left[A_k^R\right] = E\left[A_k^I\right] = 0$ and $E\left[\left(A_k^R\right)^2\right] = E\left[\left(A_k^I\right)^2\right] = \sigma^2$. Then, as $N \to \infty$, for any closed and finite interval $T \subseteq R$

$$\{s_N(t),\, t\in T\} \xrightarrow{\mathcal{D}} \{s(t),\, t \in T\}$$

where $\xrightarrow{\mathcal{D}}$ implies *convergence in distribution* and $s(t)$ is a zero-mean stationary Gaussian random process defined over the interval T, with real part $x(t)$ and imaginary part $y(t)$ such that

$$E\left[x(t_i)x(t_j)\right] = E\left[y(t_i)y(t_j)\right] = \sigma^2 \mathrm{sinc}\left(\frac{2(t_j - t_i)}{T_c}\right),$$

and

$$E\left[x(t_i)y(t_j)\right] = \sigma^2 \frac{\sin^2\left(\frac{(t_j - t_i)\pi}{T_c}\right)}{\frac{\pi(t_j - t_i)}{T_c}}.$$

for all t_i and t_j in T.

With the above rigorous theoretical justification, the modern theory of extreme values of chi-squared random processes (i.e. those corresponding to the envelope process formed from the complex Gaussian process) is then employed to arrive at simple and accurate approximations to the PMEPR distributions for the envelope of the transmitted OFDM signal. It is demonstrated through simulation that these simple and well-justified expressions are as accurate as those of [2] and [3], and, like the results in [2] and [3], apply surprisingly well for uncoded and coded OFDM systems with only a modest number of subcarriers.

This paper is organized as follows. Section 2 overviews the key parts of the proof of the convergence of the envelope process. Section 3 uses the modern theory of extreme values to derive simple approximations for the PMEPR distribution of OFDM signals and presents numerical results that demonstrate the accuracy of these expressions. Finally, Section 4 presents the conclusions.

II. The Main Theorem

In this section, the proof of the following result, which is of independent interest and captures the key properties for the proof of Theorem 2, is sketched. For the details of the proof, the reader is encouraged to obtain [5] from the authors. Theorem 2 then follows in a straightforward manner, and thus its proof is omitted here.

Theorem 1

Let

$$x_N(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \left(A_k^R \cos\left(2\pi \frac{k}{NT_c} t\right) - A_k^I \sin\left(2\pi \frac{k}{NT_c} t\right)\right) \quad (2)$$

with $t \in T$, for any closed and finite interval $T \subseteq R$, where $\{A_k^R, k = 0, \cdots, N-1\}$ and $\{A_k^I, k = 0, \cdots, N-1\}$ are each sequences of IID random variables, with $E\left[A_k^R\right] = E\left[A_k^I\right] = 0$, $E\left[\left(A_k^R\right)^2\right] = E\left[\left(A_k^I\right)^2\right] = \sigma^2$, and the two sequences are independent of each other. Then, as $N \to \infty$,

$$\{x_N(t),\, t \in T\} \xrightarrow{\mathcal{D}} \{x(t),\, t \in T\}$$

where $x(t)$ is a zero-mean stationary random process defined over T with autocorrelation function

$$E\left[x(t_i)x(t_j)\right] = \sigma^2 \mathrm{sinc}\left(\frac{2(t_j - t_i)}{T_c}\right),\ \forall t_i, t_j \in T$$

First, the framework is formalized and measurability of the appropriate quantities is given in Section 2.1. The proof of Theorem 1 then can be established by proving two key items: (1) the sequence of measures is tight, and (2) the finite dimensional distributions converge to the appropriate limiting finite dimensional distribution.

A. Framework of the Proof

In this paper, all probabilities are defined on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where Ω is the outcome space, $\mathcal{F}$ is the σ-field on Ω and $\mathcal{P}$ is the probability measure defined on $\mathcal{F}$. Let $C = C[0,1]$ be the space of continuous functions on $[0,1]$ with the uniform metric $\rho(x,y) = \sup_{t\in[0,1]} |x(t) - y(t)|, x, y \in C$ and define $\mathcal{C}$ as the σ-field of Borel sets of C. The appropriate measurability of the random functions $x_N(t)$ (in the sense that $x_N^{-1}\mathcal{C} \subset \mathcal{F}$) is then readily established [5].

A sequence $\{x_N\}$ of random function of C *converges in distribution* to the random function x, denoted by $x_N \xrightarrow{\mathcal{D}} x$, if: (1) a sequence $\{x_N\}$ of measures is tight, and (2) the finite-dimensional distributions at times $(t_1, t_2, \cdots, t_K)$ converge in distribution to $x\,(t_1, t_2, \cdots, t_K)$ for any K and $(t_1, t_2, \cdots, t_K)$.

B. Tightness

The sequence $\{x_N\}$ of random functions of C is tight if and only if these two conditions hold [6, pg. 55] :

<u>Condition 1.</u> For each positive η, there exists an a such that

$$\mathcal{P}\{|x_N(0)| > a\} \le \eta,\ N \ge 1 \quad (3)$$

<u>Condition 2.</u> For each positive ε and η, there exists a δ, with $0 < \delta < 1$, and an integer N_0 such that

$$\mathcal{P}\left\{ \sup_{\substack{|s-t|<\delta \\ s,t\in[0,1]}} |x_N(s) - x_N(t)| \ge \varepsilon \right\} \le \eta,\ N \ge N_0. \quad (4)$$

Establishing Condition 2 is the crux of the entire proof. The proofs of Condition 1 and Condition 2 are omitted; please refer to [5] for details.

Hence, for the sequence $\{x_N\}$ in (2) of random functions of C, both Condition 1 and Condition 2 are satisfied, and thus $\{x_N\}$ is tight [6, pg. 55].

C. Convergence of The Finite Dimensional Distributions

By means of the *Cramér-Wold Device* in [6, pg. 49], problems involving the convergence of random vectors in R^k can often be reduced to problems of convergence involving only ordinary random variables in R^1. That is, in R^k, a sequence of random vectors $\{\underline{\Gamma}_N\}$ converges weakly in distribution to a random vector $\underline{\Gamma}$ if and only if each linear combination of the components of $\underline{\Gamma}_N$ converges in distribution to the corresponding linear combination of the components of $\underline{\Gamma}$. The *Cramér-Wold Device* is employed to establish the result of Lemma 2.1. The proof is omitted.

Lemma 2.1

Let $x_N(t)$ be defined as in (2), and pick any L and collection of sample times $\{t_1, t_2, \ldots, t_L\}$. Then

$$\underline{\Gamma}_N = (x_N(t_1), x_N(t_2), \ldots, x_N(t_L))^T \xrightarrow{\mathcal{D}} \underline{\Gamma},$$

where $\underline{\Gamma}$ is an L-dimensional vector with jointly Gaussian components, mean vector $\underline{0}$, and covariance matrix Σ, where

$$\Sigma_{i,j} = \sigma^2 \mathrm{sinc}\left(\frac{2(t_i - t_j)}{T_c}\right) \tag{5}$$

Since a random function $\{x(t), t \in T = [0,1]\}$ of C is uniquely determined by its *finite-dimensional distributions* [6, pg. 30], there exists a unique random function x of C such that its continuous mappings $\pi_{t_1,\cdots,t_L}$ to R^L are L-dimensional Gaussian random vector with mean vector $\underline{0}$ and covariance matrix Σ as in (5). Hence, $\{x(t), t \in T\}$ is a Gaussian random process. Thus, due to the tightness of the sequence $\{x_N\}$ of random functions of C and the theorem of [6, pg. 54], it can be concluded that the sequence of real part random functions $\{x_N\}$ defined on a finite time interval T as in (2) is converging in distribution to a zero mean Gaussian random process whose autocorrelation function is given by (5). Thus, Theorem 1 is established, and it can be employed in a relatively straightforward manner (see [5]) to prove Theorem 2.

If $\omega_k = \frac{2\pi}{NT_c}\left(k - \frac{N-1}{2}\right)$ in (1), it can be shown as well that $s_N(t)$ is converging to $s(t)$, where $s(t)$ is a zero-mean stationary complex Gaussian random process defined over T with independent real and imaginary parts, each with autocorrelation function

$$\sigma^2 \mathrm{sinc}\left(\frac{(t_j - t_i)}{T_c}\right), \ \forall\, t_i,\, t_j \in T.$$

III. PMEPR Distribution of OFDM Signals by Extremal Theory

Let the complex baseband OFDM signal be expressed as:

$$\tilde{s}_N(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} A_k e^{j\frac{2\pi}{NT_c}\left(k - \frac{N-1}{2}\right)t}, \tag{6}$$

The PMEPR of $\tilde{s}_N(t)$ can be defined as [2]

$$\mathcal{P}_N \triangleq \frac{\max_{0 \le t \le NT_c} |\tilde{s}_N(t)|^2}{P_{av}}, \tag{7}$$

where $P_{av} = 2\sigma^2$.

According to the extremal theory [7][8], consider $\zeta(t)$, $\eta(t)$, $t > 0$ that are independent stationary Gaussian processes with zero mean, unit variance, and identical covariance function $r(t) = \mathrm{Cov}\,(\zeta(s), \zeta(s+t)) = \mathrm{Cov}\,(\eta(s), \eta(s+t))$, which admits the expansion

$$r(t) = 1 - \lambda\frac{t^2}{2} + o(t^2) \ \text{ as } t \to 0, \tag{8}$$

and suppose $\zeta(t)$ and $\eta(t)$ have continuously differentiable sample paths, with $\mathrm{Var}\,(\zeta'(t)) = \mathrm{Var}\,(\eta'(t)) = \lambda = -r''(0)$. Then

$$\chi^2(t) = \zeta^2(t) + \eta^2(t)$$

is a stationary $\chi^2(2)$-process with continuously differentiable sample paths. Suppose further that

$$r(t)\log(t) \to 0, \ \text{as } t \to \infty. \tag{9}$$

Then

$$\mathrm{P}\left\{\sup_{0<t\le T} \chi^2(t) \le u^2\right\} \to e^{-\tau} \tag{10}$$

if $T\mu(u) \to \tau$ as $T, u \to \infty$. Here u^2 is related to T by

$$u^2 = 2\log T + \log\log T + \log(\lambda/\pi) - 2\log\tau + o(1). \tag{11}$$

It is easy to show, based on (10) and (11), that

$$\mathrm{P}\left\{a_T\left(\max_{0\le t\le T} \chi^2(t) - b_T\right) \le x\right\} \to \exp\left(-e^{-x}\right), \ \text{as } T \to \infty\,, \tag{12}$$

for

$$a_T = 1/2,\ b_T = 2\log T + \log\log T + \log(\lambda/\pi)\,. \tag{13}$$

As suggested by Section 2, as $N \to \infty$, the complex baseband OFDM signal (6) is converging weakly to a complex Gaussian random process $\tilde{s}(t) = X(t) + jY(t)$, with

$$r(\tau) = E[X(t)X(t+\tau)] = E[Y(t)Y(t+\tau)] = \sigma^2 \mathrm{sinc}\left(\frac{\tau}{T_c}\right), \tag{14}$$

and

$$E[X(t_1)Y(t_2)] = 0, \forall t_1 \text{ and } t_2. \tag{15}$$

It is clear that $\chi^2(t) = \frac{1}{\sigma^2}\left(X^2(t) + Y^2(t)\right)$ is a $\chi^2(2)$-process. The conditions stated in (8) and (9) are satisfied by $r(\tau)$ in (14), with $\lambda = \frac{1}{3}\left(\frac{\pi}{T_c}\right)^2$. Hence, as $N \to \infty$, the cumulative distribution function of the PMEPR of the baseband OFDM signal has the following asymptotic characteristic,

$$\mathrm{P}\left\{\max_{0\le t\le T} \frac{1}{2\sigma^2}\left[X^2(t) + Y^2(t)\right] \le y\right\} = \mathrm{P}\left[\max_{0\le t\le T} \chi^2(t) \le 2y\right\} \longrightarrow \exp\left(-e^{-x}\right), \quad \text{as } T \to \infty, \tag{16}$$

where $x = (2y - b_T)\,a_T$. a_T and b_T are defined the same as in (12), with $\lambda = \frac{1}{3}\left(\frac{\pi}{T_c}\right)^2$.

Let $T = NT_c$, when the time-scale is normalized by T_c, $\lambda = \frac{\pi^2}{3}$, then the cumulative distribution function of the PMEPR $\mathcal{P}_N$ can be approximated as

$$\mathrm{P}\left\{\mathcal{P}_N \le y\right\} \cong \exp\left\{-e^{-y} N \sqrt{\frac{\pi}{3}\log N}\right\}, \tag{17}$$

for large enough N. Thus, this yields a simple result through a fully analytic derivation. In [2], the authors derived the following approximations for high order N by employing a number of approximations and the method of level-crossing rates [4]:

$$\mathrm{P}\left\{\mathcal{P}_N \le y\right\} \cong \exp\left\{-e^{-y} N \sqrt{\frac{\pi}{3} y}\right\}, \tag{18}$$

for P $\{\rho > y | \rho > \bar{r}\} \to 0$, where ρ is an arbitrary peak in one OFDM symbol (within $[0, NT_c]$) and $\bar{r}$ is a proper selection of threshold such that each positive crossing of the level $\bar{r}$ has a single positive peak that is above the level $\bar{r}$ [2]. The result of [2] is not fully analytic in the sense that it requires a number of approximations for even the asymptotic form, which can be observed to be slightly different than the actual. The comparison of (17) and (18) with the simulation results for uncoded OFDM system are shown in Figure 1 in terms of the complementary cumulative distribution function (CDF) of PMEPR. Note that both approximations are very accurate in predicting the PMEPR of the transmitted OFDM signal, but that the proposed result is better as expected.

Equation (17) is derived for uncoded systems under the assumption that the complex symbols on each subcarrier of an OFDM symbol are independent of each other. Moreover, the real and imaginary part of each complex symbol are independent of each other as well. However, in practical systems, forward-error correction coding across subcarriers is essential to deal with the effects of the multipath fading on them. The channel coding will make the assumption of the independence stated no longer valid. For a codeword of finite length, when the number of subcarriers is approaching infinity, this finite codeword can be regarded as one composite complex symbol on a set of finite number of subcarriers. Then the arguments will follow the same way as that for the uncoded system. Thus, the extreme value theory can be employed as well for coded systems to develop the CDF of the PMEPR in a coded OFDM signal.

The simulation results for an OFDM system coded with a $(2, 1, 6)$ convolutional code mapped to QPSK are shown in Figure 2. From these figures, it can be seen that (17) agrees well with the simulation results for a coded OFDM system.

IV. Conclusions

In this paper, the complex envelope of the transmitted signal in an OFDM system has been shown to converge weakly to a Gaussian random process as the number of subcarriers in the system goes to infinity. This result provides a rigorous justification for previous work that has assumed this result and employed level-crossing results of Rice to obtain expressions for the system PMEPR. In addition, modern extreme value theory has been employed to derive fully analytic and simple yet accurate approximations for the PMEPR for the transmitted signal in OFDM systems. Under mild assumptions, the extension to coded systems of the theorems contained herein should be easily established, and the simulation results are shown to agree well with (17).

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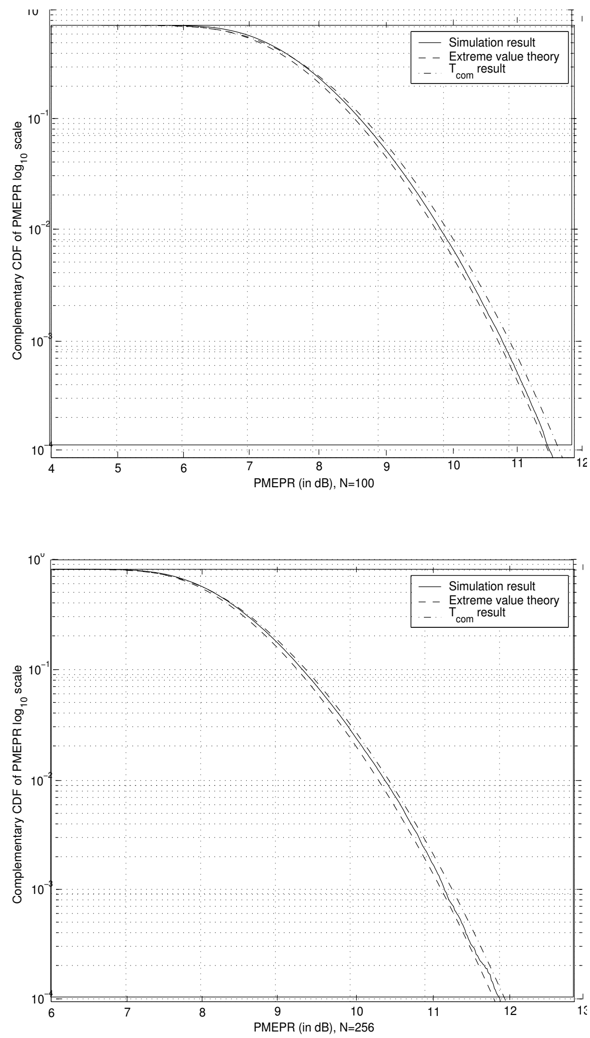

Fig. 1. Complementary cumulative distribution function of the peak-to-mean envelope power ratio (PMEPR) of an uncoded OFDM signal with 100 subcarriers (top) and 256 subcarriers (bottom) employing QPSK for the proposed expression and that of [6]. There is close agreement of the proposed expression with the simulated PMEPR for a number of subcarriers as small as 100. The simulation curves are obtained by running 10^6 independent OFDM symbols for $N = 100$, and 2.6×10^5 independent OFDM symbols for $N = 256$.

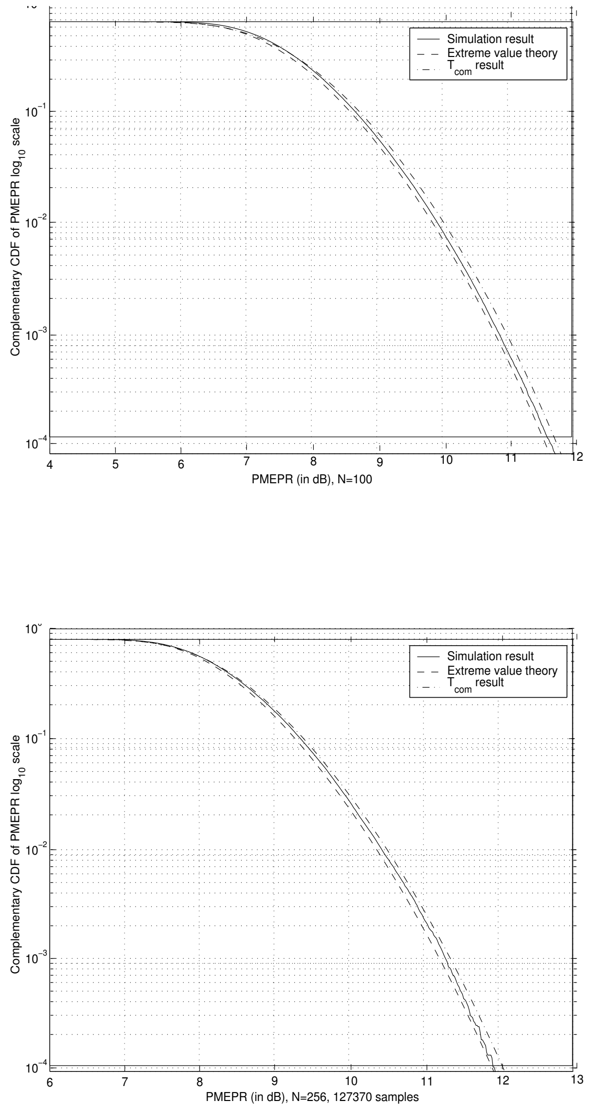

Fig. 2. Complementary cumulative distribution function (CDF) of the peak-to-mean envelope power ratio (PMEPR) of a coded, convolutional code (2,1,6), OFDM signal with 100 subcarriers and 256 subcarriers employing QPSK for the proposed expression and that of [6]. There is close agreement of the proposed expression with the simulated PMEPR for a number of subcarriers as small as 100. The closed form of the CDF of the PMEPR of an uncoded OFDM signal still holds for coded system. The simulation curves are obtained by running 10^6 independent OFDM symbols for $N = 100$ and 1.5×10^5 independent OFDM symbols for $N = 256$.