Collaboration Improves the Connectivity of Wireless Networks

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***Abstract*— In the standard approach to studying connectivity, a physical layer is assumed that allows direct transmission between neighbors within some fixed distance. The graph resulting from connecting all such pairs of neighbors reveals clusters of nodes within which communication is possible. However, future wireless networks will provide a physical layer where nodes that are connected can collaboratively search for more connections via simultaneous RF transmission and reception, thus adding connections that are not possible in the traditional non-collaborative model. The purpose of this paper is to introduce this collaborative network model and to characterize its asymptotic connectivity properties for one characterization (noncoherent power summing) of the physical layer collaboration. In the case of sparse ad hoc networks, simulations show that an infinite cluster will emerge in the infinite two-dimensional plane at a node density roughly 20% of that required in non-collaborative ad hoc networks. In the case of dense ad hoc networks, the probability for the event that the network is connected goes to one asymptotically if the transmission area of each node is no less than $\frac{4\pi(4\log N)^{\alpha/\alpha+2}(\log\log N+\log 2)^{2/\alpha+2}}{N}$, where N is the number of nodes in the network of unit area and α is the pathloss exponent. Hence, significant gains in the asymptotic connectivity properties of the ad hoc network are obtained through collaboration.**

I. Introduction

Wireless ad hoc networks are of current interest for a variety of applications, and, thus, there has been significant recent work on the connectivity and capacity of such networks. In this paper, the connectivity problem is addressed. Clearly, network connectivity is necessary for successful communication, and, hence, research on this topic is well-motivated. In addition, for networks with infrequent transmissions, the key question is whether a message can be moved across the network when necessary, hence motivating a careful study of connectivity and means of improving such - even at the potential expense of overall capacity in such networks.

Prior studies of connectivity have generally assumed the traditional layered partitioning of network functionality resulting in the characterization of the physical layer as a "bit pipe". In particular, the most common assumption is that the Boolean model characterizing the transmission between two nodes can be used to find a threshold r such that direct communication is possible for nodes separated by a distance less than r, and not possible for nodes at a distance greater than r. However, recent physical layer advances such as distributed beamforming [5] and distributed space-time coding [4] have demonstrated that relaxing the traditional layered paradigm and allowing groups of nodes to collaborate in transmission to a receiver can significantly improve performance in terms of physical layer metrics. Hence, to maximize connectivity of the network, two nodes at a separation less than r, after connecting, can collaboratively transmit to identify other nodes to which they would not connect if they transmit separately. Likewise, they can collaborate to receive transmissions simultaneously, hence making the link bidirectional in the case considered in detail in this paper.

As an example of how this might work, consider the following. Two nodes separated by a distance less than r are able to set up a direct communication. Suppose that there are no other nodes within a distance r of either of the two nodes. In the traditional non-collaborative network paradigm, the cluster would be of size two and each of these nodes would not be able to communication with any nodes outside of this cluster. Now, consider the following method of collaboration. Suppose that the two nodes simultaneously broadcast a message using a simple frequency-shift keyed (FSK) modulation format. Assuming no fine timing coordination, the two signals will add noncoherently on a multipath fading channel, hence allowing a third node to receive an average power above that required for successful decoding at a distance greater than r from the centers of these nodes. If such a third node exists, it can then send an acknowledgement signal that will be received by the two separate nodes and combined for decoding. The new cluster is of size three and can now collaborate to identify further nodes.

The goal of this paper is to characterize how such physical layer collaboration might change the connectivity properties of an ad hoc network. In particular, it could be argued that the resulting limits are the fundamental limits of connectivity in the network. Stated differently, if all network resources are used to establish connectivity, this will characterize the best possible connectivity. In this paper, two different scenarios are considered: sparse ad hoc networks, where the probability of the existence of an infinite cluster is considered, and dense ad hoc networks, where the probability that the network is

[1]This paper is based in part upon work supported by the Army Research Office under Contract DAAD10-01-1-0477 and the National Science Foundation under Grants ECS-0300130 and CCF-0430892, and employed equipment obtained under National Science Foundation Grant EIA-0080119.

completely connected is considered.

For relatively sparse networks in the infinite two-dimensional plane, previous work has focused on establishing conditions for an infinite size cluster as a function of node density, where percolation theory has been the main tool. For ad hoc networks, the most commonly employed model is continuum percolation with the Poisson Boolean model, where nodes with identical range are distributed in an infinite two-dimensional plane according to a Poisson point process. When the node density is larger than the percolation threshold (the "super-critical" case), there will be an infinite cluster almost surely. When the node density is below the percolation threshold (the "sub-critical" case), there will be no infinite cluster almost surely [7][8]. The concept of the percolation threshold was put forward over forty years ago [18], but there is still no accurate analytical expression for it. The analytical upper bound is 10.526 and the analytical lower bound is 2.195 [10][17]. Simulations have shown that the true value is around 4.5 [16]. In the first part of this work, we seek to understand whether percolation occurs in a collaborative network, and, if so, determine the value of the percolation threshold. In Section III, it is shown through simulation that the collaborative ad hoc network demonstrates percolation behavior, and, that the percolation threshold is significantly less than in non-collaborative networks.

For a dense network, we adopt the framework given by Gupta and Kumar [2][3] and focus on the connection of the entire network in a fixed area. Gupta and Kumar [2] have shown that when the covered area for each node is equal to $\frac{\log N + c(N)}{N}$, where N is the number of the nodes in the unit area disk and $\liminf_{N\to+\infty} c(N) = +\infty$, the network is completely connected with probability one as $N \to +\infty$. Otherwise, if $\limsup_{N\to+\infty} c(N) < +\infty$, there will be some isolated clusters with strictly positive probability as N increases to infinity. Therefore, the network is asymptotically disconnected with non-zero probability. Clearly, in order to maintain the connectivity of whole network, the expected number of neighbors of each node has to be on the same order of $\log N$, which goes to infinity. Naturally, we can ask a similar question for collaborative dense networks: what is the requirement of the power of each node so that the network is totally connected with probability one. This is considered in Section V, where a sufficient condition is given for a broad class of collaborative networks in the plane to be completely connected with probability one asymptotically.

Before considering the characteristics of collaborative networks in the sparse and dense cases, the models for the system are introduced in Section II. Conclusions are given in Section VI.

II. Models

A. Physical layer communication model

As noted in Section I, the development of distributed collaborative techniques allows for already connected nodes to pool resources to connect with other nodes. Clusters of nodes that would have formed under the non-collaborative model (i.e. by simply connecting all pairs of nodes within r of each other) will be denoted "first-level clusters". Nodes in a first-level cluster can then work together to connect with other nodes, thus causing smaller clusters to merge into a larger cluster, which will be termed a "second level cluster". Nodes in the second level cluster can then work collaboratively to try to link to other clusters, and, as a result, form higher level clusters. This process continues until no more connections are possibly through collaboration. The rule by which it is determined whether a given cluster can join with another cluster is considered in this section.

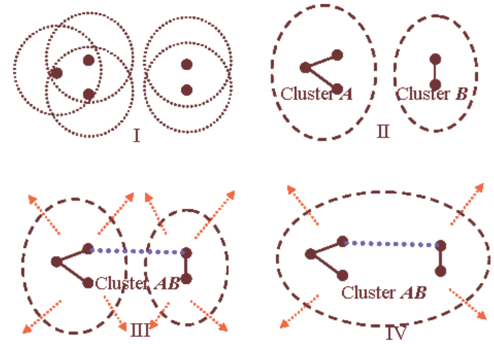

Fig. 1. Illustration of collaborative networks. I: The network with r-radius model nodes. II: Corresponding graph of non-collaborative model, two isolated clusters. One is of order 3 and the other is of order 2. III: Nodes in cluster A work collaboratively. So do the nodes in B. Thus a link over A and B emerges. IV: A second level cluster AB appears. And it works collaboratively again for expansion.

In previous studies of asymptotic connectivity, the k-neighbor model and the r-radius model have each been studied as methods to obtain the graph of connected node pairs. In the k-neighbor model, each node adjusts its power so that it can send its messages directly to its k nearest neighbors [6]. In the r-radius model, all nodes have a constant range, and two nodes within this range can communicate directly. In practice, the radius r would be obtained by determining at what range the signal power has dropped to the lowest which allows successful decoding at the receiver. In this paper, each node is assumed to be able to communicate directly with the neighbors within distance r of it. There are a number of ways that collaboration can take place on the physical layer, and, hence, a number of possible models. Some of these, such as distributed beamforming, are coherent, requiring significant signaling overhead and a rather long search time. Others, such as the FSK example given in Section I, are non-coherent, and require only rudimentary signaling overhead to establish. In this paper, the non-coherent model will be considered unless otherwise stated. Analogous to the r-radius model, the arithmetic sum of the powers received by sinks from cooperative transmitters is compared to a threshold to determine the ability of the transmitting node(s) to communicate with the receiving node(s).

B. Node distribution models

In continuum percolation studies, people usually employ the model of an infinite two-dimensional plane model, where nodes are distributed according to a Poisson point process with a given density in nodes per unit area. The notation $G_{Plane}(\lambda)$ will be used to denote the corresponding graph and network, where λ is the node density parameter, and this model will be adopted for the relatively sparse networks of Section III.

Generally, a network is called a dense network if the expectation of the number of the neighbors of each node in the network goes to infinity. Gupta and Kumar give a powerful framework within which to study the asymptotic properties of a dense network [2][3]. The distribution area can be a disk of unit area [2], a square of unit area [6], a sphere of unit area [3] or a torus of unit area [19]. We use $G_{Disk}(N)$, $G_{Square}(N)$, $G_{Sphere}(N)$, and $G_{Torus}(N)$ to indicate these, respectively, where N is the number of the nodes in the area. If the number of the nodes in the fixed area is a Poisson random variable with parameter N, we use $G^{Poisson}(N)$ to express this case. For example, $G^{Poisson}_{Disk}(N)$ represents a network distributed in disk of unit area, where the number of nodes is a Poisson random variable with parameter N. Both the unit disk and unit square models have edge effects, which are caused by the nodes on the boundary and generally lead to a tedious analysis of their effects. Perhaps surprisingly, edge effects do not greatly complicate the establishment of the sufficiency condition for collaborative networks in Section V.

III. Sparse Networks: Percolation Threshold of Collaborative Networks

First, the performance of non-collaborative networks and collaborative networks is studied in the one-dimensional case. Then, the two-dimensional case is addressed, where the percolation threshold is investigated for a collaborative network and compared to that of a non-collaborative network.

A. One-dimensional case

In the one-dimensional case $G(\lambda)$, suppose that the node communication radius is r and that there is a node X_0 at the origin. The distance L between the node X_0 and the rightmost node X_K with which X_0 can communicate is considered. Let D_j represent the distance between the node X_{j-1} and X_j. Then D_j is a exponential random variable with parameter $1/\lambda$.

In a non-collaborative network, given the condition that $j \leq K$, D_j is constrained by $\{D_j \leq r\}$. Then L is the sum of the truncated D_j, where j is from 1 to K and K has a geometric distribution $(1-e^{-r\lambda}, e^{-r\lambda})$. Hence,

$$L = \sum_{j=1}^{K} D_j \qquad D_j\text{: } j = 1, 2 \ldots K \text{ are i.i.d.} \tag{1}$$

$$\text{P}(K{=}k) = \left(1 - e^{-r\lambda}\right)^k e^{-r\lambda} \tag{2}$$

$$P(D_j \leq d_j | j \leq K) = \begin{cases} 0 & \text{if } d_j < 0 \\ \frac{1-e^{-d_j\lambda}}{1-e^{-r\lambda}} & \text{if } 0 \geq d_j \leq r \\ 1 & \text{if } d_j > 1 \end{cases} \tag{3}$$

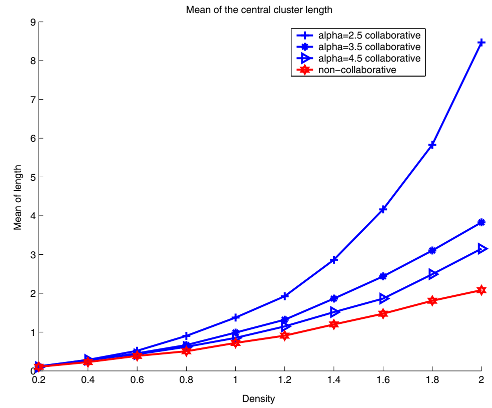

Fig. 2. Average of the distance L from node X_0 to the rightmost node of its cluster X_K given $r = 1$.

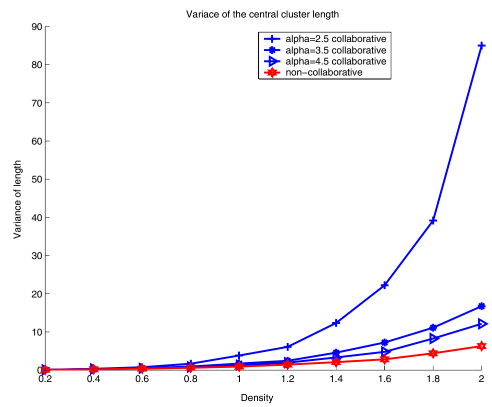

Fig. 3. Variance of the distance L from the node X_0 to the rightmost node of its cluster X_K given $r = 1$.

The first level clusters of the collaborative are generated by connecting nodes less than distance r apart (i.e. the final clusters of the non-collaborative network). The higher level clusters of the collaborative network emerge during the process of collaborative transmission and receive among the nodes that are already connected. Thus the first level cluster with node X_0 and the first level cluster on its right might be in the same higher level cluster and thus can communicate with each other via collaboration.

Simulation results are shown in Figs. 2 and 3. From these simulations, it can be observed that the improvement in the collaborative model is sensitive to the node density and pathloss exponent. The improvement is larger when the node density is larger because, in this case, the first-level clusters have a higher probability of containing multiple nodes. On the other hand, when the density is extremely low, most clusters consist of only a single node, and the collaborative case degenerates to the non-collaborative case. When the node density is a constant and the pathloss exponent (represented by α) is a variable, the performance improvement decreases with an increase in α. This is reasonable, since a larger α leads to a smaller incremental increase in area that a cluster can contact. Naturally, these trends will still apply in the more important two-dimensional case considered next.

B. Two-dimensional case

From Section III-A, the expectation of the number of nodes in a cluster for the one-dimensional non-collaborative case is finite regardless of the node density. Thus, the probability for a node to be in an infinite cluster is always zero. For the two dimensional case, the story is very different. Gilbert [18] noted that an infinite cluster appears almost surely when the density exceeds a threshold as in the case of lattice percolation. For continuum percolation, it is known that an infinite cluster appears when the expectation of the number of nodes in a cluster is infinity [7][17]. When the area of radio coverage (the "disk") of each node is unity, the analytical upper bound for the percolation threshold is 10.526 and the analytical lower bound is 2.195 [17][10]. Simulations show that the actual percolation threshold is around 4.5. Thus, if each node has, on average, more than 4.5 neighbors, the infinite cluster will appear almost surely.

For the collaborative model, a node in a multi-node cluster essentially covers an area of more than πr^2 because of the help of the others. The dependence of the coverage of a node on the location of the other nodes in the same cluster greatly complicates analysis. In particular, even showing the existence of the percolation threshold is problematic. The existence of the threshold at which the expected size of the cluster with the node at the origin is infinite is also unknown. If both of these two thresholds exist, whether they are equal is also an open problem. In this section, first steps towards the consideration of these quantities is the goal.

Here, let (X, ρ, λ) denote the Poisson Boolean model arising from an underlying Poisson point process X of density λ and radius random variable ρ, as in [7]. Per above, it is assumed that the occupation area of each node is one, so the radio radius is $\frac{1}{\sqrt{\pi}}$, yielding $(X, \frac{1}{\sqrt{\pi}}, \lambda)$. Based on Section II, analysis of the collaborative model must use received power to identify the boundary of the clusters rather than the union of the coverage of the nodes. A power threshold of unity is employed; hence, the transmission power of each node is $(\frac{1}{\sqrt{\pi}})^\alpha$.

Simulation results are conducted on a square with edge length d. Percolation is said to occur when there exists a cluster that connects the square from the top to the bottom as in [11][20]. When there exists such an infinite cluster, the ratio of its size and that of the whole network is calculated. Since a Poisson point process is ergodic (*Proposition 2.6* pp.26 [7]), this ratio is equal to the probability that an arbitrary node belongs to an infinite cluster given that one exists. The product of this ratio and the probability of the existence of an infinite cluster is the *percolation probability*, which is defined as the probability that an arbitrary node belongs to the infinite cluster [8]. As expected, Figs. 4 and 5 demonstrate that percolation occurs when λ is approximately 4.5. When the scale of the square is increased to make certain that the simulation characterizes an infinite network, the curves in Fig. 4 and 5 become steeper and intersect.

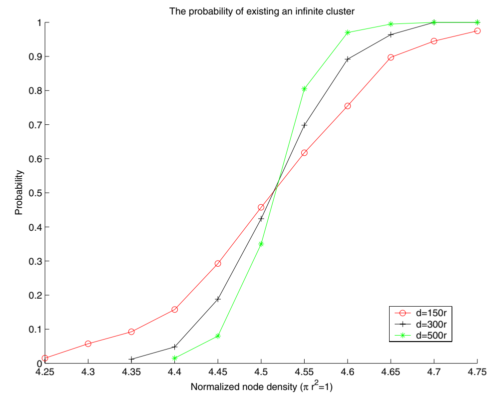

Fig. 4. Simulation for the percolation threshold of a non-collaborative network. The horizontal axis is the node density with $\pi r^2 = 1$. Nodes are distributed according to a 2D Poisson point process on a square with edge length d. Percolation occurs when there is a cluster that connects the square from the top to the bottom. The vertical axis is the probability of existing such a cluster. The percolation threshold is around 4.5.

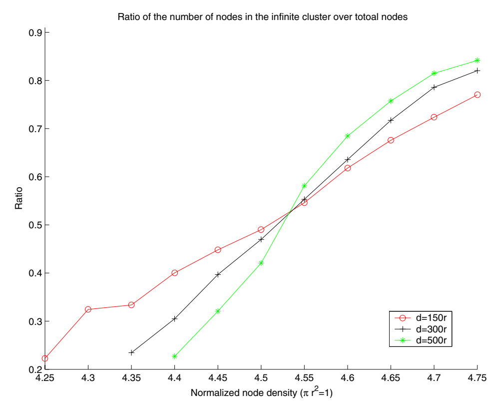

Fig. 5. Ratio between the number of the nodes in the cluster connecting top and bottom and the number of the nodes in the square given the condition that such a cluster exists in a non-collaborative network.

The simulations are repeated for a non-coherent collaborative network. A similar phenomenon of percolation threshold is observed in Fig. 6, namely the existence of a percolation threshold such that if the node density is below it, there does not exist an cluster penetrating the square from the top to the bottom, and if the node density is above it, there exists one. For pathloss exponents α of 2.0, 2.5, 3.5 and 4.5, the percolation threshold P_c is approximately 0.9, 1.3, 1.9 and 2.4 respectively. All of these are significantly less than the percolation threshold of a non-collaborative network, which is around 4.5. When we increase the scale of the square, the shifts of the curves are slight and we observe the intersection of the curves when α is 3.5 and 4.5. Given the condition that there is a cluster connecting top and bottom, this cluster contains the main part of the nodes in the network (see Fig. 7), which is quite different from a non-collaborative network.

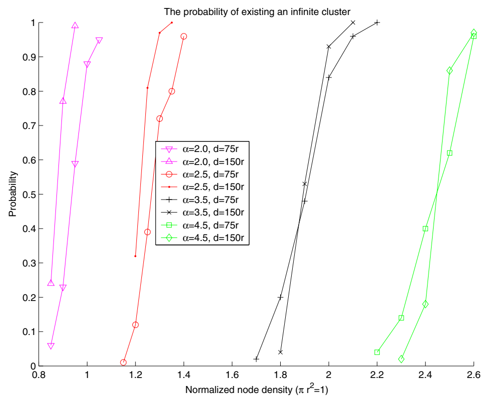

Fig. 6. Simulation for the percolation threshold of a non-coherent collaborative network. The horizontal axis is the node density with $\pi r^2 = 1$. Nodes are distributed according to a 2D Poisson point process on a square with edge length d. Percolation occurs when there is a collaborative cluster that penetrates the square from the top to the bottom. The vertical axis is the probability of existing such a cluster. P_c is sensitive to α.

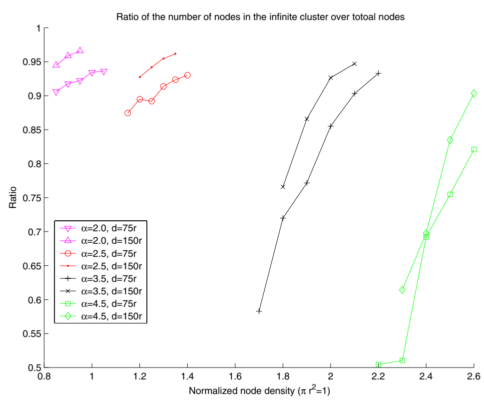

Fig. 7. Ratio between the number of the nodes in the cluster connecting top and bottom and the number of the nodes in the square given the condition that such a cluster exists in a collaborative network. When α is small, this ratio is fairly high.

IV. Dense Case: Required Power for Completely Connected Non-Collaborative Networks

As the title implies, the main result of this paper is a sufficiency result for a general class of collaborative networks in the plane presented in Section V. However, while working toward an analogous collaborative network result for the sphere model in [22], we were able to establish a technical result of interest for non-collaborative networks. This result is described in this section.

In [2], Gupta and Kumar show that the network $G_{Disk}(N)$ is completely connected with probability one as $N \to +\infty$ if and only if $\pi r^2 = \frac{\log N + c(N)}{N}$, where $\liminf c(N) = +\infty$. For the sphere model of interest here, they show in [3] (*corollary 5.1*) that the graph $G_{Sphere}(N)$ has an isolated node and is disconnected with positive probability if $\pi r^2 = \frac{\log N + c(N)}{N}$, where $\limsup c(N) < \infty$. (Note: For dense networks, recall the convention that N is used to present the total number of nodes or node density instead of λ.)

In this section, we ask what is the sufficient condition for the complete connection of a non-collaborative network $G_{Sphere}(N)$. Let C_k represent a cluster of order k, namely a cluster with k nodes. From [2], it is clear that the key factor of the proof for the sufficient condition of the connectivity of $G_{Disk}(N)$ is the C_1 domination theorem (see (4)). Naturally, this result should be extended to the sphere model before finding the sufficiency condition for $G_{Sphere}(N)$. To consider such, we first prove the C_k shrinking lemma in $G_{Sphere}(N)$, which leads to the C_k domination theorem in $G_{Sphere}(N)$ and then extends to the C_1 domination corollary in $G_{Sphere}(N)$. With this corollary, we find the sufficient condition for the connectivity of $G_{Sphere}(N)$, which is the same as that for $G_{Disk}(N)$.

The occupation area of a node X_0 is defined as the domain such that if another node is located in this domain, X_0 is able to communicate with it. Since only path loss is considered in this paper and nodes are identical, each node can communicate with the nodes within distance r of it. Therefore, the occupation area of a node is disk centered at X_0 with radius r, denoted by $B_r(X_0)$. The occupation area of a cluster is defined as the union the occupation areas of each node. Generally, the area of a $B_r(X_0)$ on the sphere is no longer πr^2 because the surface is not flat. Since r will shrink to zero in the limit, it is reasonable that local properties can be assumed when considering the characterization of finite-sized clusters. Thus, the surface effect will be neglected here and the occupation area of a node will be assumed to still be πr^2, as in a plane.

It is known that (Propositions 6.4-6.6 [7]), in $G_{Plane}^{Poisson}(\lambda, r(\lambda))$,

$$\lim_{\lambda \to +\infty} \frac{1}{q_1(\lambda, r(\lambda))} \sum_{k=1}^{\infty} q_k(\lambda, r(\lambda)) = 1 \tag{4}$$

where λ is the node density, $r(\lambda)$ is the radius of each node and $q_k(\lambda, r(\lambda))$ is the probability for a node to be in an order-k cluster. These propositions tell us that if a node is in a finite order cluster, it is in an order-1 cluster with probability one asymptotically. Let's focus on a node X_0 at a fixed point of $G_{Sphere}(N)$ and extend (4) from the infinite plane model to the sphere model. Given the condition that X_0 is in an order-$(k+1)$ cluster (denoted by C_{k+1}), what is the structure of this cluster? Is it a string, a circle, or some other topology? We will show that whatever it is, the cluster has high probability to be in the disk centered at X_0 with radius d, where $d/r \to 0$, as $N \to +\infty$.

Let $B_d(X_0)$ be the disk centered at X_0 with radius d, $P_{in}(X_0; C_{k+1}; d)$ represent the probability of the event that X_0 is in a C_{k+1} (cluster of order-(k+1)) and that the C_{k+1} is completely contained in $B_d(X_0)$. Let $P_{out}(X_0; C_{k+1}; d)$ represent the probability of the event that X_0 is in a C_{k+1} and that the C_{k+1} is not completely contained in $B_d(X_0)$. Then: $P_{in}(X_0; C_{k+1}; d) \downarrow$ as $d \downarrow$, $P_{out}(X_0; C_{k+1}; d) \uparrow$ as $d \downarrow$, and

$$P_{in}(X_0; C_{k+1}; d) + P_{out}(X_0; C_{k+1}; d) = P(X_0 \in C_{k+1}) \tag{5}$$

Lemma 1 (C_k shrinking lemma): In graph $G_{Sphere}(N, r)$, where $N \to +\infty$, $r(N) \to 0$ and $N\pi r^2(N) \to +\infty$, $\forall k \in \mathbb{N}$, $\forall \gamma \in (0,1)$, let $d = \frac{r}{(Nr^2)^\gamma}$; then,

$$\frac{P_{in}(X_0; C_{k+1}; d)}{P_{out}(X_0; C_{k+1}; d)} \to +\infty \text{ as } N \to +\infty. \tag{6}$$

Proof: First, we find a bound on $P_{out}(X_0, C_{k+1}, d)$. For any collection of k nodes $X_1, X_2, \ldots, X_k$ which potentially form a cluster of size $k+1$ with X_0, it is clear that such a cluster does not form if $X_1, X_2, \ldots, X_k$ are not completely in $B_{kr}(X_0)$. In addition, the remaining $N-k-1$ nodes must be outside the covered area of the C_{k+1}, So the feasible area for the others should be no more than $1-(\pi r^2+\sqrt{3}rd)$ (see Fig. 8). Thus, it is straightforward to show:

$$P_{out}(X_0, C_{k+1}, d) \leq \binom{N}{k} \left(k^2\pi r^2\right)^k \left[1-(\pi r^2+\sqrt{3}rd)\right]^{N-k-1} \tag{7}$$

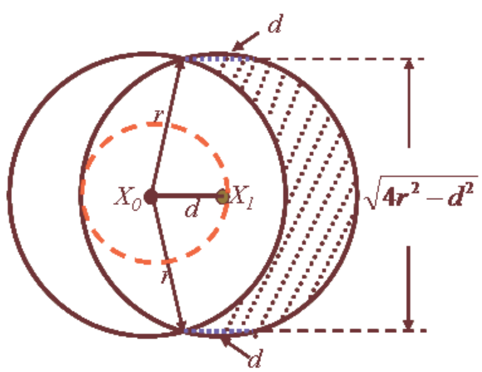

Fig. 8. The union of the shaded area and $B_r(X_0)$ is the minimal occupation area for a C_{k+1} when the C_{k+1} is not contained in $B_d(X_0)$. It is achieved when $k-1$ nodes are at the same position of X_0 and one, X_1, is on the boundary of $B_d(X_0)$. The shaded area is less than $d\sqrt{4r^2-d^2} < \sqrt{3}rd$ (without a loss of generality, $d < r/2$).

The quantity $P_{in}(X_0, C_{k+1}, d)$ can be analyzed in a similar manner. Without loss of generality, assume $d < r/2$. Clearly, if $X_1, X_2, \ldots X_k$ are contained in $B_d(X_0)$, they are all connected to one other and to X_0. Also, in order to isolate these $k+1$ nodes, the $N-k-1$ remaining nodes should be out of the covered area of the C_{k+1}, so the feasible area for the part containing the rest of the nodes should be no less than $1-\pi(r+d)^2$ (see fig 9). Thus, it is straightforward to show:

$$P_{in}(X_0, C_{k+1}, d) \geq \binom{N}{k} \left(\pi d^2\right)^k \left[1-\pi(r+d)^2\right]^{N-k-1} \tag{8}$$

Choose $0 < t < d$, such that

$$\pi(r+t)^2 \leq \pi r^2 + \sqrt{3}rd/2 \tag{9}$$

Then:

$$t \leq \sqrt{r^2+\sqrt{3}rd/2\pi} - r = \frac{\sqrt{3}rd/2\pi}{\sqrt{r^2+\sqrt{3}rd/2\pi}+r} \tag{10}$$

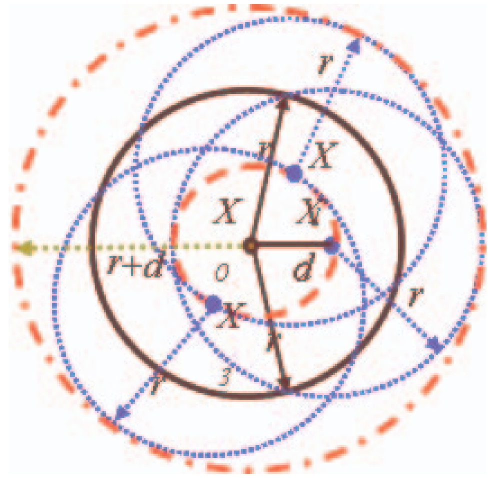

Fig. 9. The occupation area for the C_{k+1} is contained in $B_{r+d}(X_0)$ when the nodes of C_{k+1} are in $B_d(X_0)$.

Since $d < r$, we have $\sqrt{r^2+\sqrt{3}rd/2\pi} < 2r$; let

$$t = \frac{\sqrt{3}rd/2\pi}{3r} = \frac{d}{2\sqrt{3}\pi}, \tag{11}$$

t satisfies: $\pi(r+t)^2 \leq \pi r^2 + \sqrt{3}rd/2$. Observe that $d > t$ and $P_{in}(X_0, C_{k+1}, d)$ is a monotonically increasing function of d; thus,

$$\frac{P_{in}(X_0, C_{k+1}, d)}{P_{out}(X_0, C_{k+1}, d)} > \frac{P_{in}(X_0, C_{k+1}, t)}{P_{out}(X_0, C_{k+1}, d)} \tag{12}$$

$$= \frac{\binom{N}{k}\left(\pi t^2\right)^k \left[1-\pi(r+t)^2\right]^{N-k-1}}{\binom{N}{k}\left(k^2\pi r^2\right)^k \left[1-(\pi r^2+\sqrt{3}rd)\right]^{N-k-1}} \tag{13}$$

$$\geq \left(\frac{1}{12\pi^2k^2}\right)^k \left(\frac{d}{r}\right)^{2k} \frac{\left[1-(\pi r^2+\sqrt{3}rd/2)\right]^{N-k-1}}{\left[1-(\pi r^2+\sqrt{3}rd)\right]^{N-k-1}} \tag{14}$$

$$\approx \left(\frac{1}{12\pi^2k^2}\right)^k \left(\frac{d}{r}\right)^{2k} \frac{e^{-(\pi r^2+\sqrt{3}rd/2)(N-k-1)}}{e^{-(\pi r^2+\sqrt{3}rd)(N-k-1)}} \tag{15}$$

$$\approx \left(\frac{1}{12\pi^2k^2}\right)^k \left(\frac{d}{r}e^{\sqrt{3}rdN/4k}\right)^{2k} \tag{16}$$

Let $s = \frac{d}{r}$, then

$$f(d) = \frac{d}{r}e^{\sqrt{3}rdN/4k} = se^{\sqrt{3}sNr^2/4k} \tag{17}$$

$\forall \gamma \in (0,1)$, let $s = (Nr^2)^{-\gamma}$, we have:

$$f(d) = \frac{e^{\sqrt{3}(Nr^2)^{1-\gamma}/4k}}{(Nr^2)^\gamma} \to \infty \tag{18}$$

■

From Lemma 1, it is observed that in $G_{Sphere}(N)$, if a node is in a C_{k+1}, then the other k nodes are very close to it; here, d is not related to k. (Actually, the bound for d can be improved by further analysis of (16)). As imagined, the expected number of nodes in an area that is comparable to πr^2 is infinite. If the

C_{k+1} is not all in $B_d(X_0)$, it has to occupy extra area, while the probability for the existence of such extra empty area goes to zero quickly. Thus, the nodes in the C_{k+1} have to gather tightly. One of the interesting results related to d is $M_d(X_0)$, the expectation of the number of nodes in $B_d(X_0)$.

$$M_d(X_0) = N\pi d^2 = \frac{N\pi r^2}{(Nr^2)^{2\gamma}} \to 0 \text{ if } \gamma > 1/2 \quad (19)$$

This yields the following theorem:

Theorem 1 (C_k domination theorem): For a fixed node X_0 in graph $G_{Sphere}(N)$, where $N \to +\infty$, $r(N) \to 0$ and $N\pi r^2(N) \to +\infty$, $\forall k \in \mathbb{N}$,

$$\frac{P(X_0, C_{k+1})}{P(X_0, C_k)} \to 0 \text{ as } N \to +\infty \quad (20)$$

Proof: Let $\gamma \in (0.5, 1)$ and $d = \frac{r}{(Nr^2)^\gamma}$. By Lemma 1:

$$P(X_0, C_k) \approx P_{in}(X_0, C_k, d) \quad (21)$$

$$P(X_0, C_{k+1}) \approx P_{in}(X_0, C_{k+1}, d) \quad (22)$$

So the comparison of $P_{in}(X_0, C_k, d)$ and $P_{in}(X_0, C_{k+1}, d)$ can be used in place of the comparison of $P(X_0, C_k)$ and $P(X_0, C_{k+1})$. Let $I_d^k(x_0, \dots, x_k)$ or I_d^k be an indicator function that equals one when nodes at the positions $x_0, \dots, x_k$ are all in $B_d(X_0)$. Naturally:

$$I_d^k(x_0, \dots, x_k) \geq I_d^{k+1}(x_0, \dots, x_{k+1}) \quad (23)$$

Since $d \ll r$, I_d^k implies that $X_0, \dots, X_k$ are connected. Let $S_k(x_0, \dots, x_k)$ or S_k be the area covered by the nodes located at $x_0, \dots, x_k$. Note that:

$$S_k(x_0, \dots, x_k) \leq S_{k+1}(x_0, \dots, x_{k+1}) \quad (24)$$

Since N nodes are uniformly distributed on the sphere, the joint probability density function of k uniformly distributed nodes is 1. Thus we have,

$$P_{in}(X_0, C_k, d) = \binom{N}{k} \int_{x_1} \cdots \int_{x_k} I_d^k \,(1 - S_k)^{N-k-1} \, dx_1 \dots dx_k \quad (25)$$

Similarly,

$$P_{in}(X_0, C_{k+1}, d) = \binom{N}{k+1} \int_{x_1} \cdots \int_{x_{k+1}} I_d^{k+1} \,(1 - S_{k+1})^{N-k-2} \, dx_1 \dots dx_{k+1} \quad (26)$$

$$= \binom{N}{k+1} \int_{x_1} \cdots \int_{x_{k+1}} I_d^k \,(1 - S_k)^{N-k-2} \, dx_1 \dots dx_{k+1} \quad (27)$$

$$\leq \binom{N}{k+1} \int_{x_1} \cdots \int_{x_k} \pi d^2 I_d^k \,(1 - S_k)^{N-k-2} \, dx_1 \dots dx_k \quad (28)$$

By comparison of (25) and (26), when $I_d^k\,(1 - S_k) \neq 0$:

$$\frac{\binom{N}{k} I_d^k \,(1 - S_k)^{N-k-1}}{\binom{N}{k+1} \pi d^2 I_d^k \,(1 - S_k)^{N-k-2}} \approx \frac{k+1}{N\pi d^2} \to +\infty \quad (29)$$

By (21),(22) and (29):

$$\frac{P(X_0, C_{k+1})}{P(X_0, C_k)} \to 0 \quad (30)$$

■

Corresponding to (4), this theorem tells us that in a $G_{Sphere}(N)$, if a node X_0 is in a finite cluster, then as N increases to infinity, the probability for it to be in a C_k is much higher than it is in a C_{k+1}. Thus, the following corollary, which corresponds to (4), follows:

Corollary 1 (C_1 domination corollary): For a fixed node X_0 in graph $G_{Sphere}(N)$, where $N \to +\infty$, $r(N) \to 0$ and $N\pi r^2(N) \to +\infty$, $\forall K \in \mathbb{N}$

$$\lim_{N \to +\infty} \frac{\sum_{k=1}^K P(X_0, C_k)}{P(X_0, C_1)} = 1 \quad (31)$$

With the help of Theorem 1, the arguments of Section 3 in [2] can be repeated to yield the following theorem.

Theorem 2: Graph $G_{Sphere}(N)$ is asymptotically connected with probability one if $\pi r^2 = \frac{\log N + c(N)}{N}$, where $\lim c(N) = \infty$.

With Corollary 5.1 of [3], we now know that $\pi r^2 = \frac{\log N + c(N)}{N}$, where $\lim c(N) = \infty$, is the necessary and sufficient condition for the entire connection for $G_{Sphere}(N)$.

With Corollary 1, the probability of a node X_0 being in the infinite cluster can be estimated:

$$P(X_0 \in C_\infty) \approx 1 - P(X_0 \in C_1) \quad (32)$$

$$= 1 - (1 - \pi r^2)^{N-1} \quad (33)$$

$$\approx 1 - e^{-N\pi r^2} \to 1 \quad (34)$$

From the above derivations, two nodes in $G_{Sphere}(N)$ chosen from a uniform distribution are both in the giant cluster with probability one. Thus they are connected with high probability as $N \to +\infty$, implying that most of the nodes in dense networks are connected.

Lemma 2: In graph $G_{Sphere}(N)$, where $N \to +\infty$, $r(N) \to 0$ and $N\pi r^2(N) \to +\infty$, let K be the number of the clusters of order one, then

$$\sup(K) \leq \frac{7}{\pi r^2} \quad (35)$$

Proof: Let S be the union of the covered area of all the clusters of order one, obviously, $S \leq 1$. Since each point A in $\mathcal{S}^2$ can be covered by at most seven C_1 at the same time (A is the center of a C_1 and the other six C_1 have the topology of a honeycomb with edge length r.), then $S \geq \frac{K\pi r^2}{7}$. Then:

$$1 \geq \inf(S) \geq \inf\left(\frac{K\pi r^2}{7}\right) \Rightarrow$$

$$\sup(K) \leq \frac{7}{\pi r^2} \quad (36)$$

■

Since $N\pi r^2 \to +\infty$ in $G_{Sphere}(N)$, $K/N \to 0$. Thus the ratio between the number of nodes in the finite clusters and N goes to zero and a randomly chosen pair of source and destination is connected with very high probability.

One shortcoming of the $G(N)$ model of fixed area is that the node number is fixed by N. Thus the node distributions in two non-overlapping area are not independent, which generally complicates the analysis [6]. On the other hand, the $G^{Poisson}(N)$ model is free of this problem and is frequently used in the modeling and analysis of wireless networks. It motivates us to extend these results to the graph $G^{Poisson}_{Sphere}(N)$. We note that most of the properties of $G_{Sphere}(N)$ apply to $G^{Poisson}_{Sphere}(N)$, as follows.

Lemma 3: In graph $G^{Poisson}_{Sphere}(N, r)$, where $N \to +\infty$, $r(N) \to 0$ and $N\pi r^2(N) \to +\infty$, $\forall k \in \mathbb{N}$, $\forall \gamma \in (0, 1)$, let $d = \frac{r}{(Nr^2)^\gamma}$; then,

$$\frac{P_{in}(X_0; C_{k+1}; d)}{P_{out}(X_0; C_{k+1}; d)} \to +\infty \text{ as } N \to +\infty. \quad (37)$$

Theorem 3: For a fixed node X_0 in graph $G^{Poisson}_{Sphere}(N)$, where $N \to +\infty$, $r(N) \to 0$ and $N\pi r^2(N) \to +\infty$, $\forall k \in \mathbb{N}$,

$$\frac{P(X_0, C_{k+1})}{P(X_0, C_k)} \to 0 \text{ as } N \to +\infty \quad (38)$$

Corollary 2: For a fixed node X_0 in graph $G^{Poisson}_{Sphere}(N)$, where $N \to +\infty$, $r(N) \to 0$ and $N\pi r^2(N) \to +\infty$, $\forall K \in \mathbb{N}$,

$$\lim_{N\to+\infty} \frac{\sum_{k=1}^{K} P(X_0, C_k)}{P(X_0, C_1)} = 1 \quad (39)$$

Theorem 4: Graph $G^{Poisson}_{Sphere}(N)$ is asymptotically connected with probability one if and only if $\pi r^2 = \frac{\log N + c(N)}{N}$, where $\lim c(N) = \infty$.

The results above arise, because the asymptotic connectivity properties of $G^{Poisson}(N)$ in a unit area are the same as $G(N)$ in a unit area. Let Y denote the number of nodes in $G^{Poisson}(N)$ in a unit area. Then Y is a Poisson random variable with parameter N. The following lemma bounds Y.

Lemma 4: Let Y be a Poisson random variable with parameter N, then

$$\lim_{N\to+\infty} P\left(Y \in \left[N - \sqrt{\pi N/2}, N + \sqrt{\pi N/2}\right]\right) = 1 \quad (40)$$

Proof: Let

$$A(N) = 2\sqrt{\pi N/2}\ P\left(Y = N - \sqrt{\pi N/2}\right) \quad (41)$$

$$B(N) = 2\sqrt{\pi N/2}\ P\left(Y = N + \sqrt{\pi N/2}\right) \quad (42)$$

By Stirling's Approximation:

$$\lim_{N\to+\infty} A(N) = \lim_{N\to+\infty} B(N) = 1 \quad (43)$$

Then, $\forall n_1, n_2 \in [0, N]$, where $n_1 < n_2$, the following is true:

$$P(Y = n_1) \le P(Y = n_2) \quad (44)$$

Also $\forall n_1, n_2 \in [N, +\infty]$, where $n_1 < n_2$, the following is correct:

$$P(Y = n_1) \ge P(Y = n_2) \quad (45)$$

Thus,

$$P\left(Y \in \left[N - \sqrt{\pi N/2}, N + \sqrt{\pi N/2}\right]\right) \ge \min(A(N), B(N)) \quad (46)$$

Naturally,

$$\lim_{N\to+\infty} P\left(Y \in \left[N - \sqrt{\pi N/2}, N + \sqrt{\pi N/2}\right]\right) \ge \lim_{N\to+\infty} \left[\min\left(A(N), B(N)\right)\right] \to 1 \quad (47)$$

■

Thus although the number of nodes in $G^{Poisson}(N)$ is a random variable, it fluctuates around N and the amplitude of the fluctuation can be ignored in comparison with N as $N \to +\infty$. This property of the Poisson distribution leads to the similar performance of graph $G^{Poisson}(N)$ and $G(N)$.

V. Dense case: required power for completely connected collaborative networks

In a dense network, most of the nodes are in the infinite cluster. Therefore, if nodes can work collaboratively, most of the isolated clusters will be absorbed and become part of the infinite cluster. Thus, situations where a non-collaborative network is not completely connected while the corresponding collaborative network is completely connected can be expected. The main result of this paper is that such a collaboration can significantly reduce the power required for complete connectivity for any unit-area network belonging to a very general collection of networks in the plane. This result is established in this section.

Typical shapes for networks in the plane might be the unit disk, the unit square, the unit hexagon, etc. However, one can envision scenarios where an obstruction (say, a large building or vehicle) might prevent network nodes from occupying a certain area, and thus it is of interest to consider as general of a scenario as possible. Consider an arbitrarily fine gridding of the infinite plane and the restriction of such to the network shape of interest. The boundaries of the shape will cut through a number of grid squares, resulting in partial grid squares within the shape of interest along the boundary and full grid squares within the interior of the shape. What will be shown below is that the sufficiency condition for connectivity presented here will apply as long as grid squares within the shape of interest satisfy: (1) for any two full grid squares, one can travel from one square to the other by traversing only full grid squares within the shape of interest, and (2) any partial grid square touches a full grid square within the shape of interest.

Thus, in this section, consider the general class of networks in the plane that contains any unit-area shape for which: (1) the interior points form a connected set, and (2) each boundary point is arbitrarily close to an interior point. Denote an arbitrary element in this class as $G^{Poisson}_{2D}(N)$ or $G_{2D}(N)$. The following theorem is the main result.

Theorem 5: In a collaborative network $G^{Poisson}_{2D}(N)$, where the power of each node satisfies $\pi r^2 \ge \frac{4\pi(4\log N)^{\alpha/\alpha+2}(\log\log N + \log 2)^{2/\alpha+2}}{N}$, $G^{Poisson}_{2D}(N)$ is asymptotically connected with probability one.

Proof: Consider the gridding of the infinite plane into squares of edge length $\beta r/\sqrt{2}$, where $\beta >> 1$. It will be shown later that $\beta r/\sqrt{2} \to 0$ as $N \to +\infty$, and thus an

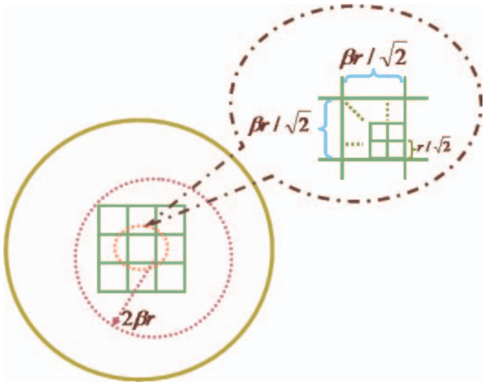

Fig. 10. Division of the surface. The unit-area shape will contain at most $2/\left(\beta^2 r^2\right)$ full grid squares with edge length $\beta r/\sqrt{2}$. Each of these squares can then be divided into β^2 sub-squares with edge length $r/\sqrt{2}$. In a collaborative network, if a square S_1 has a sub-square with more than $(2\beta)^\alpha$ nodes, the nodes in the square S_1 are connected with nodes in the 8 neighbor squares of S_1.

arbitrarily fine gridding will be achieved. The square S_2 will be termed a neighbor of square S_1 if S_2 shares at least one point with S_1. We say that a square S_1 is connected with all of its neighbors if any node X_0 in S_1 is connected with all the nodes in S_1 and S_1's neighbor squares. Similarly, we say a square S_1 is not completely connected with neighbors if there exists a pair of nodes (X_1, X_2), where X_1 is in S_1 and X_2 is in S_1 or S_1's neighbor squares that is disconnected from each other. Now, by the definition of $G_{2D}^{Poisson}(N)$, complete connectivity of the network will be achieved if, as $N \to +\infty$, every full grid square within the unit-area shape of interest is non-empty and is connected with all of its (partial or full) neighbor squares.

Each (full) square is divided into β^2 sub-squares with edge length $r/\sqrt{2}$. For each sub-square, all the nodes inside are connected because the maximum distance between any two points of the sub-square is r. If a square S_1 contains a sub-square with more than $(2\beta)^\alpha$ nodes, S_1 is connected with its neighbor squares because the power from the nodes in this sub-square at a distance $2\beta r$ is:

$$\frac{(2\beta)^\alpha r^\alpha}{(2\beta r)^\alpha} = 1 \tag{48}$$

Finally, if a square S_1 has more than $\beta^2 (2\beta)^\alpha$ nodes, then S_1 is connected with all its neighbors because there must be a sub-square within S_1 having more than $(2\beta)^\alpha$ nodes. This key idea is illustrated in Fig. 10.

On the contrary, if the network is disconnected, there must exist a full square S_k that is not completely connected with its neighbors, furthermore, there must be fewer than $\beta^2 (2\beta)^\alpha$ nodes in this square S_k. Thus, it yields:

$$\begin{aligned}
&P_{discon}\left(G_{2D}(N)\right) \\
&\leq P\left(\bigcup_{k=1}^{\frac{2}{\beta^2 r^2}} \{S_k \text{ is not completely connected with neighbors}\}\right) \\
&\leq \frac{2}{\beta^2 r^2} P\left(\{S_1 \text{ is not completely connected with neighbors}\}\right) \\
&\leq \frac{1}{\frac{r^2}{2}} P\left(\{S_1 \text{ is not completely connected with neighbors}\}\right) \qquad (49) \\
&\leq \frac{N}{\frac{Nr^2}{2}} P\left(\{S_1 \text{ is not completely connected with neighbors}\}\right) \\
&\leq NP\left(\{S_1 \text{ is not completely connected with neighbors}\}\right) \qquad (50) \\
&\leq NP\left(\{S_1 \text{ has less than } \beta^2 (2\beta)^\alpha \text{ nodes.}\}\right) \qquad (51) \\
&= N \sum_{m=0}^{\beta^2 (2\beta)^\alpha} \binom{N}{m} \left(\beta^2 r^2/2\right)^m \left(1 - \beta^2 r^2/2\right)^{N-m} \qquad (52) \\
&= N \sum_{m=0}^{m_\alpha} \binom{N}{m} s_\beta^m \left(1 - s_\beta\right)^{N-m} \qquad (53)
\end{aligned}$$

where $m_\alpha = \beta^2 (2\beta)^\alpha$ and $s_\beta = \beta^2 r^2/2$. The eventual choice of β will go to infinity in the following derivatives, which leads to (49). (50) is valid because the dense network is studied in this section; therefore, $N\pi r^2 \to +\infty$. Now, let's switch to $G_{2D}^{Poisson}(N)$:

$$\begin{aligned}
&P_{discon}(G_{2D}^{Poisson}(N)) \\
&= \sum_{j=1}^{\infty} P_{discon}(G_{2D}(j)) e^{-N} \frac{N^j}{j!} \qquad (54) \\
&\leq \sum_{j=1}^{\infty} \left\{ e^{-N} \frac{N^j}{j!} j \sum_{m=0}^{\min(m_\alpha, j)} \binom{j}{m} s_\beta^m (1 - s_\beta)^{j-m} \right\} \qquad (55) \\
&= \sum_{m=0}^{m_\alpha} \sum_{j=m}^{\infty} \left\{ j \binom{j}{m} e^{-N} \frac{N^j}{j!} s_\beta^m (1 - s_\beta)^{j-m} \right\} \qquad (56)
\end{aligned}$$

Focus on one term of the equation above:

$$A_m = \sum_{j=m}^{\infty}\left\{j\binom{j}{m}e^{-N}\frac{N^j}{j!}s_\beta^m(1-s_\beta)^{j-m}\right\} \quad (57)$$

$$= \frac{s_\beta^m N^m e^{-N}}{m!}\sum_{j=m}^{\infty}\frac{j\left\{N(1-s_\beta)\right\}^{j-m}}{(j-m)!} \quad (58)$$

$$= \frac{s_\beta^m N^m e^{-N}}{m!}\left\{\sum_{j=m+1}^{\infty}\frac{j\left\{N(1-s_\beta)\right\}^{j-m}}{(j-m)!}+m\right\}$$

$$\leq \frac{s_\beta^m N^m e^{-N}}{m!}\left\{\sum_{j=m+1}^{\infty}\frac{(m+1)\left\{N(1-s_\beta)\right\}^{j-m}}{(j-m-1)!}+m\right\}$$

$$\leq \frac{s_\beta^m N^{m+1} e^{-N}}{m!}\left\{\sum_{j=0}^{\infty}\frac{(m+1)\left\{N(1-s_\beta)\right\}^{j}}{j!}+\frac{m}{N(1-s_\beta)}\right\}$$

$$\approx \frac{m+1}{m!}s_\beta^m N^{m+1}e^{-N}e^{N(1-s_\beta)} \quad (59)$$

$$= \frac{(m+1)N}{m!}(Ns_\beta)^m e^{-Ns_\beta} \quad (60)$$

Assume that $Ns_\beta > m_\alpha$, that is

$$N\beta^2 r^2/2 > \beta^2\left(2\beta\right)^\alpha \Leftrightarrow \pi r^2 > \frac{2\pi\left(2\beta\right)^\alpha}{N} \quad (61)$$

then A_m, $m \in \{0,1,2,\ldots,m_\alpha\}$ is an increasing function of m:

$$P_{discon}(G_{2D}^{Poisson}(N)) \leq m_\alpha A_{m_\alpha} \quad (62)$$

$$= m_\alpha\frac{(m_\alpha+1)N}{m_\alpha!}(Ns_\beta)^{m_\alpha}e^{-Ns_\beta} \quad (63)$$

Thus, a proper configuration of r and β so that (63) goes to 0 is desired. First note that β is not a parameter of the network but rather a parameter used to divide the plane. Thus, any value of β can be chosen as long as $\beta \gg 1$ and $\beta r/\sqrt{2} \to 0$. For example, we might let $Ns_\beta = 2\log N$, choose β so that $(Ns_\beta)^{m_\alpha}$ goes to infinity with the speed of N, and choose r so that the assumption is satisfied. Thus (63) will go to zero as $N \to +\infty$ because N, $m_\alpha!$, $(Ns_\beta)^{m_\alpha}$ are canceled and the remaining goes to zero. We will find that such a configuration satisfies all of the assumptions. Choosing β such that $(2\log N)^{m_\alpha} = N$

$$\beta = \left(\frac{\log N}{2^\alpha\left(\log 2 + \log\log N\right)}\right)^{1/(\alpha+2)} \quad (64)$$

Thus, $\beta \gg 1$ is satisfied as $N \to \infty$. Then, letting $Ns_\beta = 2\log N$, we have

$$\pi r^2 = \frac{4\pi\log N}{N\beta^2} \quad (65)$$

$$= \frac{4\pi\left(4\log N\right)^{\alpha/\alpha+2}\left(\log\log N+\log 2\right)^{2/\alpha+2}}{N} \quad (66)$$

$$= O\left(\frac{2\pi\left(2\beta\right)^\alpha}{N}\right) \text{ as } N \to +\infty \quad (67)$$

Thus, (61) and assumption $\beta r/\sqrt{2} \to 0$ are guaranteed. Obviously, when $(2\log N)^{m_\alpha} = N$ and $Ns_\beta = 2\log N$,

$$P_{discon}(G_{2D}^{Poisson}(N)) < m_\alpha\frac{(m_\alpha+1)N}{m_\alpha!}(Ns_\beta)^{m_\alpha}e^{-Ns_\beta} \quad (68)$$

$$= \frac{m_\alpha+1}{(m_\alpha-1)!} \to 0 \quad (69)$$

It means that the network will be asymptotically connected with probability one when (66) is satisfied. ■

A similar conclusion for $G_{2D}(N)$ can be derived by the same method. Let consider an expression for the upper bound of $P_{discon}(G_{2D}(N))$, by (53),

$$P_{discon}(G_{2D}(N)) \leq N\sum_{m=0}^{m_\alpha}\binom{N}{m}s_\beta^m(1-s_\beta)^{N-m} \quad (70)$$

$$\leq \sum_{m=0}^{m_\alpha}\frac{N}{m!}(Ns_\beta)^m(1-s_\beta)^{N-m} \quad (71)$$

By assuming that $Ns_\beta > m_\alpha$, $m_\alpha s_\beta \to 0$, we have:

$$P_{discon}(G_{2D}(N)) \leq \frac{(1-s_\beta)^{-m_\alpha}}{(m_\alpha-1)!}N\left(Ns_\beta\right)^{m_\alpha}\left(1-s_\beta\right)^N \quad (72)$$

$$\approx \frac{e^{m_\alpha s_\beta}}{(m_\alpha-1)!}N\left(Ns_\beta\right)^{m_\alpha}e^{-Ns_\beta} \quad (73)$$

Then when (64) and (66) hold, the assumptions are satisfied and

$$P_{discon}(G_{2D}(N)) \leq \frac{e^{m_\alpha s_\beta}}{(m_\alpha-1)!} \to 0 \quad (74)$$

Thus we have:

Theorem 6: In a collaborative network $G_{2D}(N)$, where the power of each node satisfies $\pi r^2 \geq \frac{4\pi(4\log N)^{\alpha/\alpha+2}(\log\log N+\log 2)^{2/\alpha+2}}{N}$, then $G_{2D}(N)$ is asymptotically connected with probability one.

Theorems 5 and 6 are sufficient conditions for a completely connected collaborative network. Recalling the results from [2] the gain caused by collaboration is at least on the order of $(\frac{\log N}{\log\log N})^{2/\alpha+2}$.

Recall the necessity conclusion from [2]: when the power of the node equals $\frac{\log N + c(N)}{N}$ and $\limsup c(N) < +\infty$, there are some clusters of order one with positive probability. In the dense collaborative networks, whether there will be a threshold for power under which clusters of order one will appear with positive probability is unknown. Consider an isolated node X_0 in the non-collaborative network $G_{Disk}^{Poisson}(N)$. If it is assumed that all of the other nodes are connected in the corresponding collaborative network, the mean of the collaborative power at node X_0 goes to infinity as $N \to +\infty$. Thus, it seems reasonable that most order-1 clusters will be absorbed. Whether some of them will remain with non-zero probability after the collaboration is still an open problem.

VI. CONCLUSION

Collaboration at the physical layer is emerging as a powerful tool in ad hoc networks. In this paper, a first investigation of the difficult characterization of connectivity with the aid of collaborating nodes has been undertaken. For a network employing noncoherent collaboration (or power summing), which is the most likely form of collaboration due to its implementation simplicity, connectivity results have been established for both sparse and dense networks. In the sparse case, simulations suggest that the probability of the existence of an infinite cluster in a collaborative network exhibits a percolation threshold, which is significantly less than the percolation threshold observed in a non-collaborative network.

For the dense case, a more analytical study of the probability of when the collaborative network is completely connected has been presented. In a traditional non-collaborative network, only a small portion of the nodes are in clusters containing a finite number of nodes. In the limit, it has been shown that any of these finite order clusters must be of very small (vanishing) size. With the help of this result, the C_1 domination theory for the two-dimensional plane is successfully extended to the sphere model. Hence, it has been established that the results in [2] are still valid for the unit area sphere.

Finally, it has been established that collaborative networks require less power than non-collaborative networks in order to maintain connectivity of the whole network. The sufficient condition for the connectivity of collaborative networks is $\frac{4\pi(4\log N)^{\alpha/\alpha+2}(\log\log N+\log 2)^{2/\alpha+2}}{N}$. Generally, the expectation of the collaborative signal power at a C_1 goes to infinity in any dense networks. It is natural to guess that any dense network is completely connected, but the proof of this statement is an open problem.

ACKNOWLEDGMENT

Part of the this work was done while the first two authors were visiting the Laboratory for Information and Decisions Systems (LIDS) at the Massachusetts Institute of Technology. The authors wish to thank Professor Muriel Medard and her research group for fruitful conversations on this topic.

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