ECE 603 - Probability and Random Processes, Fall 2009

Homework #4

Due: 11/06/09, in class

1. Let X be a Gaussian random variable with mean $\mu = 3$ and variance $\sigma^2 = 1$. Where needed, use the $\text{erf}(\cdot)$ function, as defined in class.

   (a) Sketch $f_X(x)$, the probability density function of X.

   (b) Find the probability that $-4 \leq X \leq 1$.

   (c) Find the probability that $X^2 \geq 10$.

   (d) Find $F_X(x|A) = P(\{X \leq x\} \bigcap A)/P(A)$, that is, find the cumulative distribution function of X given that the event A has occurred, where $A = \{X \geq 4\}$. Use it to find $f_X(x|A)$.

2. You have a table that gives you the value of the "Goeckel Function" for all $x \geq 0$:

$$\mathcal{G}(x) = \int_x^\infty \frac{1}{2} e^{\frac{-u^2}{5}} du$$

   Let Y be a Gaussian random variable with mean μ and variance σ^2; that is, $Y \sim N(\mu, \sigma^2)$. Write an expression for $P(Y \leq y)$ for all y in terms of $\mathcal{G}(x)$.

3. The random variables X and Y have joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} cx^2y^2, & 0 \leq x \leq 1, x \leq |y| \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

   (a) Find the value of c.

   (b) Write an expression (no need to evaluate) for $P(Y < 1 - X)$.

   (c) Find $f_{X|Y}(x|y)$, the conditional probability density function of X given $Y = y$. **For your limits (which you should not forget), use $-1 \leq y \leq 1$ and then bound x in terms of y.**

   (d) Are X and Y independent?

   (e) Somebody tells you that $Y = 0.5$. Given this information, find the x_0 such that $P(|X - x_0| < 0.1)$ is maximized.

4. Suppose that X and Y are **independent** random variables with probability density functions:

$$f_X(x) = \begin{cases} 2x, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} 3y^2, & 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find $E[XY]$.

(b) Find cov(X, Y).

(c) Find $P(X > Y)$.

(d) Find $E[X|\{X > Y\}]$.

5. The marginal density function of X is given by:

$$f_X(x) = \begin{cases} 0.75(2x - x^2), & 0 \leq x \leq 2 \\ 0 & \text{else} \end{cases}$$

We also know that, given $X = x$, the random variable Y is uniformly distributed on $[0, x]$.

Note: Being very careful with your limits is the key to this problem.

(a) Find $f_{Y|X}(y|x)$.

(b) Find the joint probability density function $f_{X,Y}(x, y)$.

(c) Find the marginal probability density function $f_Y(y)$.

6. *Jointly Gaussian random variables:*

(a) Let X and Y be jointly Gaussian random variables. Let $E[X] = 0$, $E[Y] = 0$, $E[X^2] = 4$, $E[Y^2] = 4$, and $\rho_{X,Y} = \frac{1}{3}$. Define $Z = 2X - 3Y$. Find $P(Z > 3)$.

(b) Let X and Y be jointly Gaussian random variables. Suppose you know that $E[X] = 0$ and $E[Y] = 0$, and that X and Y are independent. By doing measurements, you find that $E[X^2Y^2] = 4$, and that $E[X^2 + Y^2] = 5$. Find possible values for the pair (σ_X^2, σ_Y^2), the variances of X and Y, respectively, **and (do not forget this part)** the correlation coefficient $\rho_{X,Y}$.