ECE 603 - Probability and Random Processes, Fall 2009

Homework #3

Due: 10/19/09, in class

1. A number is chosen at random from the interval [0,1]. As is the standard case, the probabilities are defined on the Borel σ-algebra (restricted to [0,1]). **Starting from first principles** (i.e. definition of the Borel σ-algebra, axioms of probability, etc.), answer the following three parts:

   (a) Let A be a subset of $[0,1]$ that is **not** in the Borel σ-algebra. Show that A must contain an uncountable number of elements.

   (b) Let D be an arbitrary uncountable subset of $[0,1]$. Is $\overline{D}$, the complement of D, necessarily countable?

   (c) Let C be the set of irrational numbers in $[0,1]$; that is, $x \in C$ if and only if $0 \leq x \leq 1$ and $x \neq \frac{m}{n}$ for all $m \in \{0,1,2,3,\ldots\}$, $n \in \{0,1,2,3,\ldots\}$. Find the probability of C.

2. A number is chosen from the interval [0,1] such that the likelihood of a given result x is proportional to the value x. Define a non-trivial probability space for this experiment; that is, find $(\Omega, \mathcal{A}, P)$, where Ω is the observation space, $\mathcal{A}$ is a set of subsets of Ω to which probabilities are assigned, and P is a probability mapping from $\mathcal{A}$ to $[0,1]$.

3. I have analyzed two *independent* experiments, Experiment 1 and Experiment 2, to arrive at two separate probability spaces: $(\Omega_1, \mathcal{A}_1, P_1)$ and $(\Omega_2, \mathcal{A}_2, P_2)$, where:

   $\Omega_1 = \{1,2\}$, $\mathcal{A}_1 = \{\phi, \{1\}, \{2\}, \{1,2\}\}$, and $P_1(\cdot)$ is defined by $P_1(\phi) = 0$, $P_1(\{1\}) = 0.4$, $P_1(\{2\}) = 0.6$, $P_1(\Omega) = 1$.

   $\Omega_2 = \{2,4\}$, $\mathcal{A}_2 = \{\phi, \{2\}, \{4\}, \{2,4\}\}$, and $P_2(\cdot)$ is defined by $P_2(\phi) = 0$, $P_2(\{2\}) = 0.2$, $P_2(\{4\}) = 0.8$, $P_2(\Omega) = 1$.

   (a) Are these valid probability spaces? *Be sure to tell me all of the conditions that you checked to arrive at your answer.*

   (b) My boss asks me to define a combined experiment as follows: Perform Experiment 1 andremember the result; then, perform Experiment 2 and remember the result. Now, write in your notebook the *outcome* of the combined experiment as an ordered pair with the first entry equal to the result of Experiment 1 and and the second entry equal to the result of Experiment 2. For example, an outcome might be "(1,4)". Find $(\Omega, \mathcal{A}, P)$ for the combined experiment. Use a $\mathcal{A}$ that captures as many events as possible, and be sure to write out explicitly at least half of the events in $\mathcal{A}$.

   (c) Alas, the boss is fickle and changes his mind. Now he asks: perform Experiment 1 and remember the result; then, perform Experiment 2 and remember the result. Now, write in your notebook the *outcome* of the combined experiment as a *random variable* X equal to the result of Experiment 1 times the result of Experiment 2. For example, an outcome might be "4" (which is 2×2). Find $(\Omega, \mathcal{A}, P)$ for the random variable X. *Hint: Feel free to define P using the integral of a function if this makes it easier to represent.*

Now, your buddy in the modeling department comes to you with yet another experiment description: $(\Omega_3, \mathcal{A}_3, P_3)$, where

$\Omega_3 = \{1,2,3\}$, $\mathcal{A}_3 = \{\phi, \{1\}, \{2,3\}, \{1,2,3\}\}$, and $P_3(\cdot)$ is defined by $P(\phi) = 0, P(\{1\}) = 0.1, P(\{2,3\}) = 0.9, P(\{1,2,3\}) = 1$.

(d) Is $(\Omega_3, \mathcal{A}_3, P_3)$ a valid probability space? *Be sure to tell me all of the conditions that you checked to arrive at your answer.*

(e) Your boss asks you to use the description of $(\Omega_3, \mathcal{A}_3, P_3)$ to find the probability that a "3" is observed. How do you respond?

4. The probability density function of a random variable X is given by $f_X(x)$, where:

$$f_X(x) = \begin{cases} cx^2, & -2 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the value of the constant c.

(b) Find the probability that $X^2 \geq 1$.

(c) Find the probability that $X - 1 \geq -\frac{1}{4}$.

(d) Let the random variable Y be defined by:

$$Y = \begin{cases} -X, & X \leq 0 \\ 0, & X \geq 0 \end{cases}$$

Find the probability density function of Y.

5. Suppose that I am observing a network connection that is good ("G") with probability 0.9 and bad ("B") with probability 0.1. Let T (in seconds) be the time until the first packet arrives.

If the connection is good, T has probability density function:

$$f_G(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0, & \text{else} \end{cases}$$

If the connection is bad, T has probability density function:

$$f_B(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & \text{else} \end{cases}$$

(a) What is the probability that the first packet arrives during the first second?

(b) Given that the first packet arrives during the first second, what is the probability that the link is good?

6. A salesman visits one of three cities: X-ville, Y-ville, and Z-ville. When he visits a given city, the corresponding probability density function (pdf) of the money that he obtains is given by:

$$
\begin{aligned}
f_X(x) &= \frac{1}{4}\,\delta(x) + \frac{1}{2}\,\delta(x-5) + \frac{1}{4}\,\delta(x-10) \\
f_Y(y) &= \frac{1}{2}\,\delta(y) + \frac{1}{2}\,\delta(y-10) \\
f_Z(z) &= \begin{cases} 1/10, & 0 \le z \le 10 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$

(a) Suppose he chooses a city at random to visit. Find the probability that he makes greater than or equal to \$5.

(b) Suppose he chooses a city at random to visit. Given that he makes greater than or equal to \$5, find the probability that he visited city X.

(c) Any of the cities can claim that they are the "best" city for the salesman to obtain money - if they use the correct argument. Give the argument that each can make. In other words, for each city, give a measure by which it is the "best".

(d) Suppose he does 20 visits to city Y, and the money obtained for each visit is indepedent of any other visit. Write an expression for the probability that he makes more than \$150. (*Your expression should only contain simple terms that are easily evaluated.*)