ECE 603 - Probability and Random Processes, Fall 2009

Homework #2

Due: 10/9/09, in class

1. (a) Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two distinct algebras for a sample space S. Is $\mathcal{A}_1 \bigcap \mathcal{A}_2$ an algebra for the sample space S? Justify your answer with either a proof or a counterexample.

   (b) Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two distinct algebras for a sample space S. Is $\mathcal{A}_1 \bigcup \mathcal{A}_2$ an algebra for the sample space S? Justify your answer with either a proof or a counterexample.

2. I flip a fair coin twice (assume the flips are independent) and record the outcome of these flips in order. For example, for "head followed by head", the outcome is "HH". Your job is to define a probability model that will used by one of your co-workers to analyze the experiment. You are not aware of what questions he/she might want to ask, so you want to generate as complete a model as possible (e.g. you do not want to use a trivial $\mathcal{A}$). Provide a probability space $(S, \mathcal{A}, P)$ for this experiment. Since the size of the sets involved here is not that large, be explicit about how each of these three things are defined. In particular, write out all of the sets in $\mathcal{A}$ and give the probability of each.

3. Your buddy who works in the modeling department provides you with the following probability space: $S = (0, 1)$, $\mathcal{A} = \mathcal{B}$ (where the Borel field is restricted to $(0, 1)$, of course) and, for any interval,

$$P((a,b)) = \begin{cases} \frac{b^2-a^2}{2}, & 0 \leq a < b \leq \frac{1}{2} \\ \frac{b^2-a^2}{2} & \frac{1}{2} \leq a < b \leq 1 \\ \frac{1}{2} + \frac{b^2-a^2}{2}, & a < \frac{1}{2} < b \leq 1 \end{cases}$$

   (a) Find the probability of the outcome $\frac{1}{4}$.

   (b) Find the probability of the set of irrational numbers in $(0, 1)$.

4. Tell whether the following statements are "True" or "False". *If you answer "True", prove the result. If you answer "False", give a counterexample.*

   (a) If the events A and B are *independent* with $P(A) = 0.1$ and $P(B) = 0.1$, then the events A and B *cannot* be disjoint (i.e. mutually exclusive).

   (b) If the events C and D are *independent* with $P(C) = 0.1$ and $P(D) = 0.0$, then the events C and D *cannot* be disjoint (i.e. mutually exclusive).

   (c) If the two events E and F are *independent*, then the events $\overline{E}$ and $\overline{F}$ are independent.

   (d) If the two events G and H are *mutually exclusive* (i.e. disjoint), then it must be the case that $\overline{G}$ and $\overline{H}$ are mutually exclusive (i.e. disjoint).

   (e) For events J and K, if we know that $P(J) \leq 0.1$, then $P(J|K)$ must be ≤ 0.1.

   (f) For events L and M, if we know that $P(L) \leq 0.1$, then $P(L \bigcap M)$ must be ≤ 0.1.

5. Suppose that A, B, and C are events such that $P[A] = P[B] = 0.3, P[C] = 0.55, P[A \bigcap B] = 0, P[\overline{A} \bigcap \overline{B} \bigcap \overline{C}] = 0.1$, and $P[A \bigcap \overline{C}] = 0.2$. Evaluate the following probabilities: *(Hint: Draw the Venn Diagram. You may then want to identify the probabilities of each of the disjoint regions in the diagram before starting the problem.)*

   (a) At least one of the events A, B, or C occurs.

   (b) Exactly one of A, B, or C occurs.

   (c) At most one of A, B, or C occurs.

   (d) C occurs, but neither A nor B occurs.

6. A bag consist of 3 coins: c_1, c_2, and c_3. The coins are unfair: c_1 has a probability of "heads" of 0.3, c_2 has a probability of heads of 0.5, and c_3 has a probability of heads of 0.6. A coin is drawn from the bag *at random* and that coin is tossed three times.

   (a) What is the probability that "heads" shows on all three tosses.

   (b) Given that "heads" shows on all three tosses, what is the probability that the coin being tossed is c_3?

   (c) Given that "tails" shows on all three tosses, what is the probability that the coin being tossed was c_1 or c_2?