ECE 745 - Advanced Communication Theory, Spring 2009

Final Exam

Take-Home (24 Hours)

Procedure

1. Pick up the exam outside my door anytime between Tuesday, May 12th, 9am, and Sunday, May 17th, noon. Sign and date the envelope.

2. **There are seven problems for 140 points.**

3. For the exam, you may use the course textbook *Elements of Information Theory,* by Cover and Thomas, and your course notes. For anything from the text, you must establish any missing steps not done in the course notes. For example, if you use the result of a theorem or homework problem, you must provide all of the steps between the course notes and the result you want to use. **No other references are allowed, including the WWW. You may not consult with anybody else for any reason - even for an issue of clarification. Instead, send me e-mail.**

4. Within 24 hours of picking up the exam, return it to one of the two following places: (1) If the Marcus 215 complex is open, slide it under the door of my office (Marcus 215H). (2) If the Marcus 215 complex is not open, place it in an envelope with "Goeckel" written prominently on the outside. Slide the envelope under the door of the "ECE Mailroom" next to Marcus 210.

Testmanship

- **Full credit will be given only to fully justified answers.**
- Giving the steps along the way to the answer will not only earn full credit but also maximize the partial credit should you stumble or get stuck. If you get stuck, attempt to neatly define your approach to the problem and why you are stuck.
- If part of a problem depends on a previous part that you are unable to solve, explain the method for doing the current part, and, if possible, give the answer in terms of the quantities of the previous part that you are unable to obtain.
- Start each problem on a new page. Not only will this facilitate grading but also make it easier for you to jump back and forth between problems.
- If you get to the end of the problem and realize that your answer must be wrong (e.g. the Fourier Transform of a real signal that is not conjugate symmetric), be sure to write "this must be wrong because . . . " so that I will know you recognized such a fact.
- Academic dishonesty **will** be dealt with **harshly -** the *minimum penalty* will be an "F" for the course.

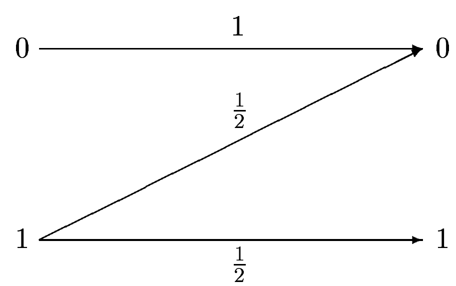
1. (a) Describe a pair of random variables X and Y such that there exists y_1 and y_2 such that

$$H(X|Y = y_1) \leq H(X) \leq H(X|Y = y_2)$$

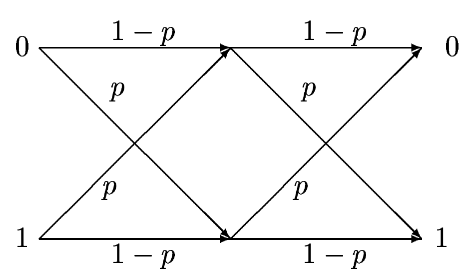
(b) Describe a pair of random variables X and Y for which $I(X;Y) = H(X) < H(Y)$.

(c) Let $Z = X + Y$ for real-valued discrete random variables X and Y. Show that $H(Z) \leq H(X) + H(Y)$. Give an example where $H(Z) = H(X) + H(Y)$ and another where $H(Z) < H(X) + H(Y)$.

2. Find the capacity of the following channel



3. Recall the binary symmetric channel with crossover probability p. Suppose that I hook two such channels together *with no encoding or decoding between the channels;* that is, the output from one channel is the input to the next channel:

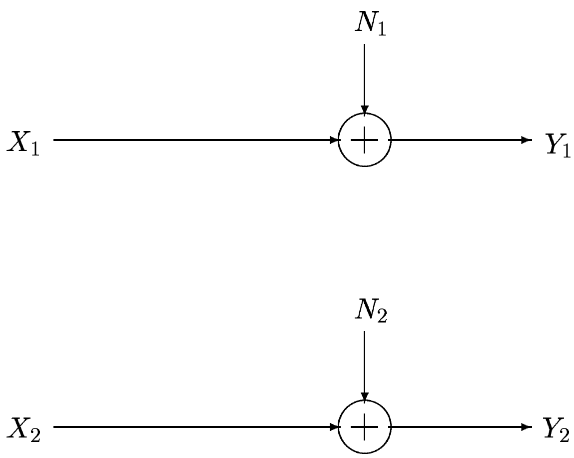


   (a) The result of the above is another binary symmetric channel (BSC) with a modified cross-over probability. Find its capacity as a function of p.

   (b) Now, suppose I hook N binary symmetric channels with crossover probability p together *with no encoding or decoding between the channels*. Find the capacity as a function of p and N. **What happens as $N \to \infty$?**.

   (c) Suppose that I again hook N binary symmetric channels with crossover probability p together, but I allow infinitely complex encoding and decoding before and after each of the binary symmetric channels. Find the capacity from the input to the output.

4. Consider a parallel Gaussian channel:



with zero-mean independent and identically distributed Gaussian noise sequences N_1 and N_2 of respective variances σ_1^2 and σ_2^2, where $\sigma_1^2 > \sigma_2^2$. Consider the power constraint $E[X_1^2 + X_2^2] \le 2P$. For the optimal solution, at what power P does the channel stop behaving like a single channel with noise σ_2^2 and begin behaving like a pair of channels?

5. For the discrete-time additive white Gaussian noise channel multiple-access channel, prove, using typical sequences, that any rate pair R_1, R_2 can be achieved as long as:

$$\begin{aligned} R_1 &< \frac{1}{2}\log\left(1+\frac{P_1}{N}\right) \\ R_2 &< \frac{1}{2}\log\left(1+\frac{P_2}{N}\right) \\ R_1+R_2 &< \frac{1}{2}\log\left(1+\frac{P_1+P_2}{N}\right) \end{aligned}$$

Give the codebook constructions, encoding and decoding rules, and show that the probability of error goes to zero.

6. Let $(X_i) \in \{0,1\}$ be an independent and identically distributed (IID) sequence of random variables, with $P(X_i = 1) = p$ and $P(X_i = 0) = 1 - p$. Let $(Z_i) \in \{0,1\}$ be an independent and identically distributed (IID) sequence, with $P(Z_i = 1) = r$ and $P(Z_i = 0) = 1 - r$. Let $Y_i = X_i \oplus Z_i$, where $\oplus$ denotes modulo-2 addition.

(a) At what rate pairs (R_1, R_2) can (X_i) and (Y_i) be separately encoded with arbitrarily small error probabiliy; in other words, what is the Slepian-Wolf region.

(b) Sketch the rate region in (a) for the special cases:

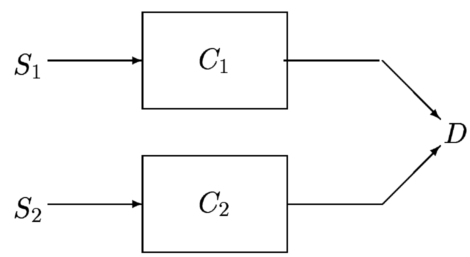
- $p = \frac{1}{2}, r = 0$
- $p = \frac{1}{2}, r = \frac{1}{2}$

7. Before starting this problem, recall the following couple of lines for bounding the error probability during the achievability part of the derivation of the Channel Coding Theorem for discrete memoryless channels (DMCs):

$$
\begin{aligned}
P_e &< \frac{\delta}{2} + \sum_{i=1}^{2^{NR}-1} 2^{-N(I(X;Y)-3\epsilon)} \\
&< \frac{\delta}{2} + 2^{NR} 2^{-N(I(X;Y)-3\epsilon)},
\end{aligned}
$$

where the first term upper bounds the probability that the received sequence is not jointly typical with the transmitted codeword, and the second term upper bounds the probability that the received sequence is jointly typical with another codeword. At the time, we were not concerned what happens when we got an error, but here we are interested in such. In particular, for $R > C$, we see that we have, on average, $2^{N(R-I(X;Y)+3\epsilon)}$ other codewords (other than the one transmitted) that are jointly typical with Y.

Now, suppose that we have the following situation. Two sources S_1 and S_2, with **non-interfering channels** with capacity C_1 and C_2, respectively, to a common destination D, desire to send a common message W (i.e. a message that they both know) of rate $R > C_1$ to the destination.



Obviously, if $R < C_1 + C_2$, the senders can split the message to accomplish reliable communication. But, alas, Sender 1 insists on sending at rate $R > C_1$. Thus, it is your job as Sender 2 to come up with a scheme that transmits at the same time and uses **no** feedback from the destination (i.e. there is no way to know which "other codewords" are jointly typical with the output of channel 1 and are thus being confused with the correct one) to clear up the confusion. Give your scheme, prove it works, and indicate how big C_2 is required to be for your scheme. (*Hint: Think about how many bits you would need to send to appropriately reduce the second term in the error probability above, and how you might set up a scheme to accomplish such.*)