ECE 603 - Probability and Random Processes, Fall 2006

Homework #6

Due 11/06/06

1. Give an example where random variables X and Y are each individually Gaussian, but X and Y not jointly Gaussian.

2. Let $X_1, X_2, \ldots, X_N$ be random variables with the same mean μ and with covariance function:

$$\text{cov}(X_i, X_j) = \sigma^2 \rho^{|i-j|}$$

where $|\rho| < 1$. Find an expression for the mean and variance of the *sample mean*:

$$\hat{\mu} = \frac{X_1 + X_2 + \ldots + X_n}{n}$$

3. Let $X_1, X_2, X_3, \ldots$ be independent random variables, each with density function $f_X(x)$ with unknown mean $E[X] = \mu$ and unknown variance $\sigma^2 = E[X^2] - \mu^2$. We estimate the sample mean as:

$$\hat{\mu} = \frac{1}{N} \sum_{i=1}^{N} X_i$$

After we have calculated the sample mean, the sample variance can be found as:

$$s^2 = \frac{1}{N-1} \sum_{i=1}^{N} (X_i - \hat{\mu})^2 \tag{1}$$

In this problem, you will show why the latter equation is used instead of the more obvious choice.

(a) Show that:

$$\sum_{i=1}^{N} (X_i - \mu)^2 = \sum_{i=1}^{N} (X_i - \hat{\mu})^2 + N(\hat{\mu} - \mu)^2.$$

(b) Use the result of part (a) to show that:

$$E\left[k \sum_{i=1}^{N} (X_i - \hat{\mu})^2\right] = k(N-1)\sigma^2$$

(c) Use part (b) to show that $E[s^2] = \sigma^2$. When the expected value of an estimator is equal to what we are trying to estimate, it is called an <u>unbiased</u> estimator.

(d) Show that if $(N-1)$ is replaced by N in (1), the estimator becomes biased.

4. An experiment is defined by the probability space $(\Omega, \mathcal{A}, P)$, where $\Omega = [0,1]$, $\mathcal{A}$ is the Borel σ-algebra restricted to $[0,1]$, and $P(\cdot)$ is defined by $P((a,b)) = b - a$. Let $X_n(\omega) = \omega^n, \omega \in \Omega$. Determine whether or not the sequence $\{X_n\}$ converges to $X = 0$ for each of the following cases. **Do the parts in order and be sure to justify each answer.**

   (a) In distribution.

   (b) In probability.

   (c) In quadratic mean.

   (d) Almost surely.

5. Consider the sequence of random variables $\{X_n\}$ for which:

$$P(X_n = 0) = 1 - \frac{1}{n}$$
$$P(X_n = 1) = \frac{1}{n}$$

   for $n = 1, 2, 3, \ldots$. Determine whether or not the sequence $\{X_n\}$ converges to $X = 0$ for each of the following cases. **Do the parts in order and be sure to justify each answer.**

   (a) In distribution.

   (b) In probability.

   (c) In quadratic mean.

   (d) Because the mapping from $(\Omega, \mathcal{A}, P)$ is not given explicitly, it is not possible to determine whether the sequence converges almost surely to $X = 0$. It could be true or not:

   - Give an $(\Omega, \mathcal{A}, P)$ and definition of $\{X_n\}$ such that:

$$P(X_n = 0) = 1 - \frac{1}{n}$$
$$P(X_n = 1) = \frac{1}{n}$$

     for which $\{X_n\}$ **does** converge almost surely to $X = 0$.
   - Give an $(\Omega, \mathcal{A}, P)$ and definition of $\{X_n\}$ such that:

$$P(X_n = 0) = 1 - \frac{1}{n}$$
$$P(X_n = 1) = \frac{1}{n}$$

     for which $\{X_n\}$ **does not** converge almost surely to $X = 0$.

6. Consider the sequence of random variables $\{X_n\}$, where X_n is a random variable uniformly distributed between 0 and $\frac{1}{n}$. Determine whether or not the sequence $\{X_n\}$ converges (and, if so, to what) for each of the following cases. **Do the parts in order and be sure to justify each answer.**

   (a) In distribution.

   (b) In probability.

   (c) In quadratic mean.

   (d) Almost surely.