ECE 603 - Probability and Random Processes, Fall 2006

Homework #1

Due 9/22/06

1. Determine whether the following sets are countable or uncountable:

   (a) The set A of all functions $f : \{0, 1\} \longrightarrow \mathcal{Z}_+$

   (b) The set B_n of all functions $f : \{1, 2, \ldots, n\} \longrightarrow \mathcal{Z}_+$

   (c) The set $C = \bigcup_{n \in \mathcal{Z}_+} B_n$.

   (d) The set D of all functions $f : \mathcal{Z}_+ \longrightarrow \{0, 1\}$.

   (e) The set E of all functions $f : \mathcal{Z}_+ \longrightarrow \mathcal{Z}_+$.

   (f) The set F of all two-element subsets of $\mathcal{Z}_+$.

   (g) The set G of all finite subsets of $\mathcal{Z}_+$.

2. Using the three axioms of probability and basic set theory, show that:

   (a) $P(\phi) = 0$

   (b) $P(A \bigcap \overline{B}) = P(A) - P(A \bigcap B)$

   (c) $P(A) = 1 - P(\overline{A})$

3. Define the set $A \oplus B = \{x | x \in A \text{ or } x \in B, \text{ but } x \text{ not in both } A \text{ and } B\}$. Using the axioms of probability and set theory, show:

   (a) $P(A \oplus B) = P(A \bigcap \overline{B}) + P(\overline{A} \bigcap B)$

   (b) $P(A \oplus B) = P(A) + P(B) - 2P(A \bigcap B)$

4. (a) Let $S = \{1, 2, 3, 4, 5\}$. Find the smallest σ-algebra $\mathcal{A}$ that contains the sets $\{1, 2, 3\}$ and $\{2, 3, 4\}$.

   (b) Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two distinct algebras for a sample space S. Is $\mathcal{A}_1 \bigcap \mathcal{A}_2$ an algebra for the sample space S? Justify your answer with either a proof or a counterexample.

   (c) Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two distinct algebras for a sample space S. Is $\mathcal{A}_1 \bigcup \mathcal{A}_2$ an algebra for the sample space S? Justify your answer with either a proof or a counterexample.

5. From class, we know that a set of subsets $\mathcal{A}$ of S is an algebra if both of the following are true:

   - If $A \in \mathcal{A}$, then $\overline{A} \in \mathcal{A}$.
   - If $A \in \mathcal{A}$, $B \in \mathcal{A}$ then $A \bigcup B \in \mathcal{A}$.

   Show that this implies that:

   (a) If $A \in \mathcal{A}$, $B \in \mathcal{A}$, then $A \bigcap B \in \mathcal{A}$.

   (b) Both ϕ and S are in $\mathcal{A}$.