Homework #3 Solutions

ECE 603 Fall, 2009

1) (a)

This can be done by showing that any countable set A is in $\mathcal{B}$.

Recall any singleton $\{x\} \in \mathcal{B}$ because

$$\{x\} = \bigcap_{n=1}^{\infty} (x - 1/n, x + 1/n)$$

Since any countable A is a countable union of singletons, $\mathcal{B}$ a σ-algebra $\Rightarrow A \in \mathcal{B}$. Thus, if $A \notin \mathcal{B}$, it must be uncountable.

(b)

No. Let $D = [0, 1/2)$. Then $\bar{D} = [1/2, 1]$ and both are uncountable.

(c) Let Q = set of rationals between 0 and 1

Since Q is countable,

$$P(Q) = P\left(\bigcup_{x \in Q} \{x\}\right) \overset{\text{Third Axiom}}{=} \sum_{x \in Q} P(\{x\})$$

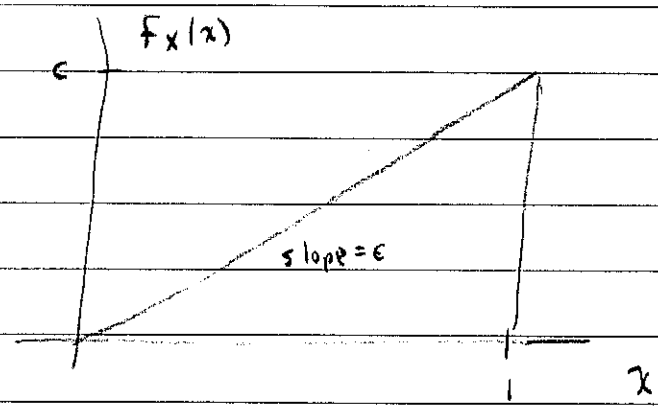
Since $P(\{x\}) = 0$ for any $\{x\}$, $P(Q) = 0$.

$\Rightarrow C = \bar{Q} \Rightarrow P(C) = 1 - P(Q) = 1.$

2)

Let $\Omega = [0,1]$, $\mathcal{a}$ = the Borel sets restricted to $[0,1]$, which are generated from intervals (a,b).

Now, we need to define the probability measure on an interval (a,b), $0 \le a \le b \le 1$. We can use the density function to do so:



$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1 \Rightarrow c = 2$$

$$\Rightarrow f_X(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 0, & \text{else} \end{cases}$$

and the probability of interval (a,b) is

$$\int_a^b 2x\,dx = x^2 \Big|_a^b = b^2 - a^2$$

thus completing

$$(\Omega, \mathcal{a}, P)$$

3) (a) Yes. Key checks: (i) $\mathcal{A}$ is a σ-algebra
(ii) $P(\cdot)$ satisfies 3 axioms

(b) $\Omega = \{(1,2),(1,4),(2,2),(2,4)\}$

$$\mathcal{A} = \mathcal{P}(\Omega) = \{\phi, \{(1,2)\}, \{(1,4)\}, \{(2,2)\}, \{(2,4)\}, \{(1,2),(1,4)\}, \{(1,2),(1,4)\}, \dots, \Omega\}$$

P:
$$\left.\begin{aligned} P((1,2)) &= 0.4\cdot 0.2 = 0.08 \\ P((1,4)) &= 0.4\cdot 0.8 = 0.32 \\ P((2,2)) &= 0.6\cdot 0.2 = 0.12 \\ P((2,4)) &= 0.6\cdot 0.8 = 0.48 \end{aligned}\right\} \text{all singletons}$$

For $A \in \mathcal{A}$, $P(A) = \sum_{x_i \in A} \underbrace{P(\{x_i\})}_{\text{sum probs of singletons in } A}$

(c)

$\Omega = \mathbb{R}$ $\quad \mathcal{A} = \mathcal{B}$

Note there are three possible outcomes

$$P(X=2) = 0.4\cdot 0.2 = 0.08$$
$$P(X=4) = 0.4\cdot 0.8 + 0.6\cdot 0.2 = 0.44$$
$$P(X=8) = 0.6\cdot 0.8 = 0.48$$

$$P((a,b)) = \int_a^b \big(0.08\,\delta(x-2) + 0.44\,\delta(x-4) + 0.48\,\delta(x-8)\big)\,dx$$

(d) Yes. See (a).

(e) not in $\mathcal{A} \Rightarrow$ <u>undefined</u>

4)

(a) $\int_{-2}^{1} cx^2\,dx = \tfrac{1}{3}cx^3\Big|_{-2}^{1} = \tfrac{c}{3}(1+8) = 3c \Rightarrow c = \tfrac{1}{3}$

(b) $P(X^2 \geq 1) = P(X \leq -1 \text{ or } X \geq 1)$

$= \int_{-2}^{-1} x^2/3 = x^3/9 \Big|_{-2}^{-1} = -1/9 + 8/9$

$= 7/9$

(c)

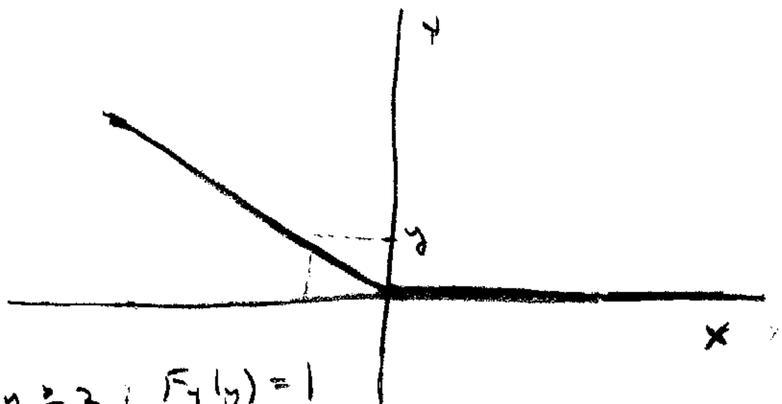
$P(X-1 \geq -1/4) = P(X \geq 3/4)$

$= \int_{3/4}^{1} x^2/3\,dx$

$= x^3/9 \Big|_{3/4}^{1} = 1/9 - (3/4)^3/9$

$= \dfrac{1-(3/4)^3}{9}$

(d)



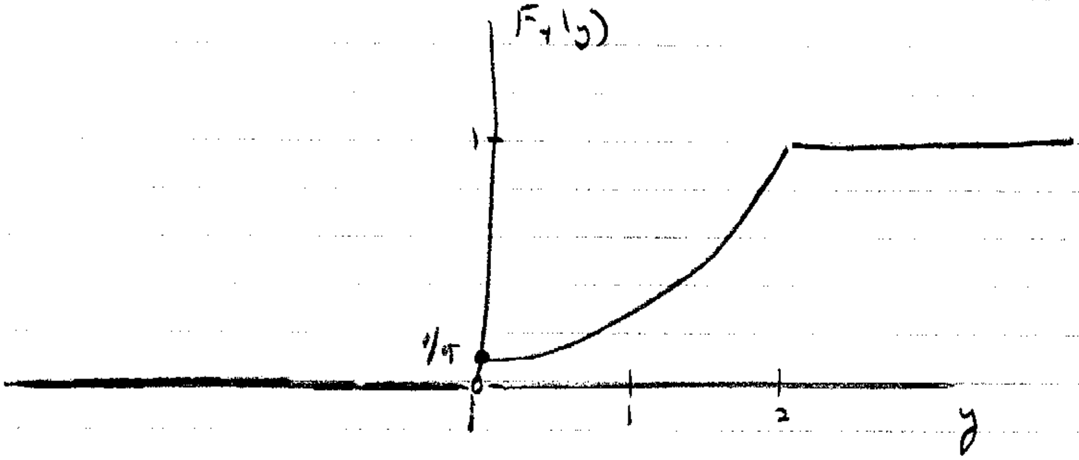
<u>For $y \geq 2$</u> : $F_Y(y) = 1$

<u>For $y < 0$:</u> $F_Y(y) = 0$

<u>For $y = 0$</u> : $P(Y=0) = P(X \geq 0) = \int_0^1 x^2/3 = 1/9$

<u>For $2 \geq y \geq 0$</u> : $F_Y(y) = P(Y \leq y) = P(-X \leq y) = P(X \geq -y) = 1/9 + \int_{-y}^{0} x^2/3\,dx$

$= 1/9 + x^3/9 \Big|_{-y}^{0} = 1/9 + y^3/9$



$$f_Y(y) = \tfrac{1}{9}\,\delta(y) + \begin{cases} 0, & y \le 0, \\ y^2/3, & 0 \le y \le 2 \\ 0, & y \ge 2 \end{cases}$$

5)

(a) $P(T \le 1) = P(T \le 1 \mid G)\,P(G) + P(T \le 1 \mid B)\,P(B)$

$$= \int_0^1 2e^{-2x}\,dx \cdot 0.9 + \int_0^1 e^{-x}\,dx \cdot 0.1$$

$$= -e^{-2x}\Big|_0^1 \cdot 0.9 - e^{-x}\Big|_0^1 \cdot 0.1$$

$$= 0.9\,(1 - e^{-2}) + 0.1\,(1 - e^{-1})$$

$$= 1 - 0.9\,e^{-2} - 0.1\,e^{-1}$$

(b) $$P(G \mid T \le 1) = \frac{P(T \le 1 \mid G)\,P(G)}{P(T \le 1)}$$

$$= \frac{0.9 - 0.9e^{-2}}{1 - 0.9e^{-2} - 0.1e^{-1}}$$

6)

(a) B: event he makes $\geq$ \$5

A_X: event he visits city X
A_Y: event he visits city Y
A_Z: event he visits city Z
$\Big\}$ partition!

$$P(B) = P(B|A_X)P(A_X) + P(B|A_Y)P(A_Y) + P(B|A_Z)P(A_Z)$$
$$= 3/4 \cdot 1/3 + 1/2 \cdot 1/3 + 1/2 \cdot 1/3 = 7/12$$

(b) $$P(A_X|B) = \frac{P(B|A_X)P(A_X)}{P(B)} = \frac{3/4 \cdot 1/3}{7/12} = 3/7$$

(c) lots of possible answers

City X: Best chance of $\geq$ \$5

City Y: Best chance of \$10

City Z: Probability 0 of coming home with \$0.

(d)

He needs to be "successful" on at least 16 visits, 50% chance of success each time.

$$\sum_{\ell=16}^{20} \binom{20}{\ell} (0.5)^{\ell} (0.5)^{20-\ell} = (0.5)^{20} \sum_{\ell=16}^{20} \binom{20}{\ell}$$