ECE 603 Fall, 2009

1) Yes

Let $A \in \mathcal{A}_1 \cap \mathcal{A}_2$

$\Rightarrow A \in \mathcal{A}_1, A \in \mathcal{A}_2$

$\Rightarrow \bar{A} \in \mathcal{A}_1, \bar{A} \in \mathcal{A}_2$ ($\mathcal{A}_1, \mathcal{A}_2$ are algebras)

$\Rightarrow \bar{A} \in \mathcal{A}_1 \cap \mathcal{A}_2$ (closed under complementation)

Let $A, B \in \mathcal{A}_1 \cap \mathcal{A}_2$

$\Rightarrow A, B \in \mathcal{A}_1, A, B \in \mathcal{A}_2$

$\Rightarrow A \cup B \in \mathcal{A}_1, A \cup B \in \mathcal{A}_2$ ($\mathcal{A}_1, \mathcal{A}_2$ are algebras)

$\Rightarrow A \cup B \in \mathcal{A}_1 \cap \mathcal{A}_2$ (closed under unions)

(b)

No

Let $S = \{1, 2, 3, 4, 5\}$

$\mathcal{A}_1 = \{\phi, S, \{1,2\}, \{3,4,5\}\}$

$\mathcal{A}_2 = \{\phi, S, \{1,2,3\}, \{4,5\}\}$

both are algebras

Now, $\mathcal{A}_1 \cup \mathcal{A}_2 = \{\phi, S, \{1,2\}, \{1,2,3\}, \{4,5\}, \{3,4,5\}\}$

and $\{1,2\} \cup \{4,5\} = \{1,2,4,5\} \notin \mathcal{A}_1 \cup \mathcal{A}_2$

not closed under unions.

2) $S = \{HH, TH, HT, TT\}$

	P
$\mathcal{F} = \{\phi,$	0
$\{HH\}$	1/4
$\{TH\}$	1/4
$\{HT\}$	1/4
$\{TT\}$	1/4
$\{HH, TH\}$	1/2
$\{HH, HT\}$	1/2
$\{HH, TT\}$	1/2
$\{TH, HT\}$	1/2
$\{TH, TT\}$	1/2
$\{HT, TT\}$	1/2
$\{TH, HT, TT\}$	3/4
$\{HH, HT, TT\}$	3/4
$\{HH, TH, TT\}$	3/4
$\{HH, TH, HT\}$	3/4
$S\}$	1

3) (a)

$$P(\{1/4\}) = P\left(\bigcap_{n=1}^{\infty} \{1/4 - 1/n, 1/4 + 1/n\}\right)$$

$$= \lim_{n\to\infty} P((1/4 - 1/n, 1/4 + 1/n))$$

$$= \lim_{n\to\infty} \frac{(1/4 + 1/n)^2 - (1/4 - 1/n)^2}{2} \quad (n \text{ large enough})$$

$$= \lim_{n\to\infty} 1/2n = 0$$

(b)

For any $x \neq 1/2$, $P(\{x\}) = 0$ from above

For $x = 1/2$: $P\left(\bigcap_{n=1}^{\infty} (1/2 - 1/n, 1/2 + 1/n)\right)$

$$= \lim_{n\to\infty}\left(1/2 + \frac{(1/2 + 1/n)^2 - (1/2 - 1/n)^2}{2}\right) = 1/2$$

$\Rightarrow P(\mathbb{Q}) = 1/2$ and $P(\overline{\mathbb{Q}}) = 1/2$

4)

(a) True

$$P(A \cap B) \overset{A,B \text{ independent}}{=} P(A)P(B) = 0.1 \cdot 0.1 = 0.01$$

$$\Rightarrow A \cap B \neq \emptyset$$

$\Rightarrow$ A, B not mutually exclusive

(b) False

Let $(\Omega, \mathcal{A}, P)$ be given by

$([0,1], \mathbb{B}, P)$ with $P((a,b)) = b - a$

$C = [0, 0.1]$ and $D = \emptyset$ is a counterexample

(c) True

$$P(\bar{E} \cap \bar{F}) = 1 - P(E \cup F)$$

$$= 1 - P(E) - P(F) + P(E \cap F)$$

$$\overset{E, F \text{ independent}}{=} 1 - P(E) - P(F) + P(E)P(F)$$

$$= (1 - P(E))(1 - P(F))$$

$$= P(\bar{E})P(\bar{F})$$

(d) False $([0,1], \mathbb{B}, P)$ with $P((a,b)) = b - a$

$G = [0, 1/4]$, $H = [1/4, 1/2]$ is a counterexample

(e)

<u>False</u>

$([0,1], \mathbb{B}, P)$ with $P((a,b)) = b-a$

$J = [0, 0.1)$ and $K = [0, 0.5)$ is a counterexample

(f) <u>True</u>

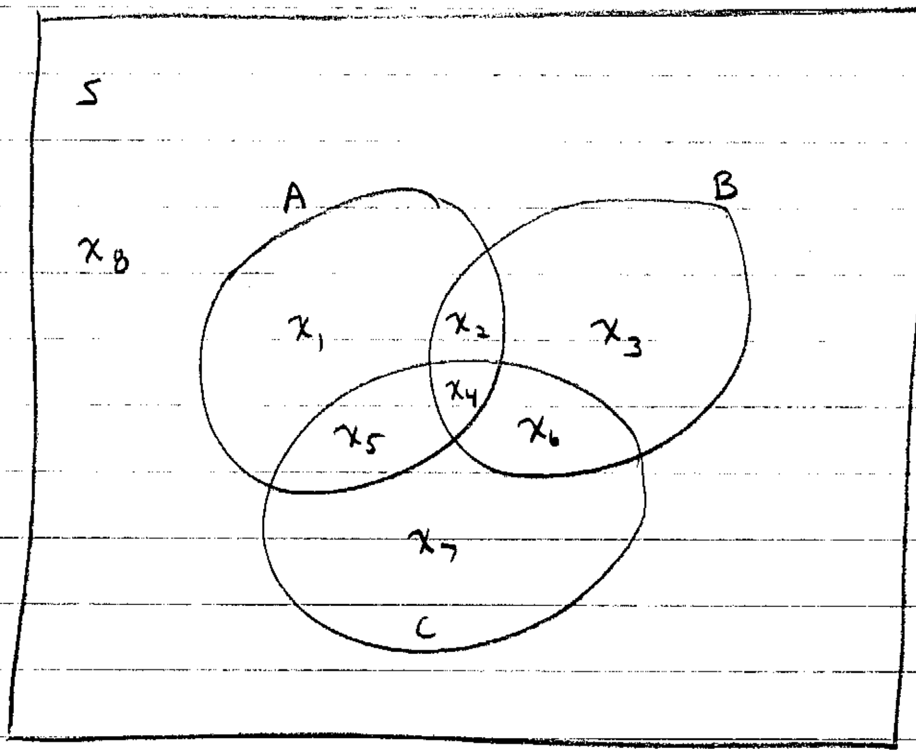
disjoint

$$L = (L \cap M) \cup (L \cap \bar{M})$$

$$\Rightarrow P(L) \overset{3^{rd}\text{ axiom}}{=} P(L \cap M) + P(L \cap \bar{M})$$

$$\Rightarrow P(L) \geq P(L \cap M)$$

5)



Let x_i = probability of region in Figure above

$(i = 1, 2, 3, \ldots, 8)$

Note that the 8 regions are disjoint. Thus, using Axiom 3, we can get equations for the x_i's.

(continued)

$$P[A \cap B] = 0 \Rightarrow x_2 + x_4 = 0$$

$$\Rightarrow x_2 = 0, \ x_4 = 0$$

$$P[A \cap \bar{C}] = 0.2 \Rightarrow x_1 + x_2 = 0.2$$

$$\Rightarrow x_1 = 0.2$$

$$P[A] = 0.3 \Rightarrow x_1 + x_2 + x_4 + x_5 = 0.3$$

$$\Rightarrow x_5 = 0.1$$

$$P[(\bar{A} \cap \bar{B} \cap \bar{C})] = 0.1 \Rightarrow x_8 = 0.1$$

$$P[B] = 0.3 \Rightarrow x_2 + x_3 + x_4 + x_6 = 0.3$$

$$\Rightarrow x_3 + x_6 = 0.3 \leftarrow$$

$$P[C] = 0.55 \Rightarrow x_4 + x_5 + x_6 + x_7 = 0.55$$

$$\Rightarrow x_6 + x_7 = 0.45 \leftarrow$$

3 equations
3 unknown!

Need one more!

$$P[S] = 1 \Rightarrow x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 1.0$$

$$\Rightarrow x_3 + x_6 + x_7 = 0.6 \leftarrow$$

$$\Rightarrow x_3 = 0.15, \ x_6 = 0.15, \ x_7 = 0.30$$

Now it is easy to get all of these - add the probabilities of disjoint sets.

(a) $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 0.9$

(b) $x_1 + x_3 + x_7 = 0.65$

(c) $x_1 + x_3 + x_7 + x_8 = 0.75$

(d) $x_7 = 0.30$

6)

H_i: heads on toss i

C_i: coin i is drawn

(a)

Recognize C_1, C_2, and C_3 form a partition.

Law of Total Prob

$$P(H_1 \cap H_2 \cap H_3) = P(H_1 \cap H_2 \cap H_3 | C_1) P(C_1) + P(H_1 \cap H_2 \cap H_3 | C_2) P(C_2) + P(H_1 \cap H_2 \cap H_3 | C_3) P(C_3)$$

$$= 0.3^3 \cdot 1/3 + 0.5^3 \cdot 1/3 + 0.6^3 \cdot 1/3$$

$$= 0.123$$

Bayes Rule

(b) $$P(C_3 | H_1 \cap H_2 \cap H_3) = \frac{P(H_1 \cap H_2 \cap H_3 | C_3) P(C_3)}{P(H_1 \cap H_2 \cap H_3)}$$

$$= \frac{0.6^3 \cdot 1/3}{0.123} = 0.585$$

same as (a)

(c) $$P(T_1 \cap T_2 \cap T_3) = 0.7^3 \cdot 1/3 + 0.5^3 \cdot 1/3 + 0.4^3 \cdot 1/3$$

$$= 0.177$$

$$P(C_3 | T_1 \cap T_2 \cap T_3) = \frac{P(T_1 \cap T_2 \cap T_3 | C_3) P(C_3)}{P(T_1 \cap T_2 \cap T_3)}$$

$$= \frac{0.4^3 \cdot 1/3}{0.177} = 0.121$$

$$\Rightarrow P(C_1 \cup C_2 | T_1 \cap T_2 \cap T_3) = 1 - P(C_3 | T_1 \cap T_2 \cap T_3)$$

$$= 0.879$$