

# Homework #1 Solutions

- 1 -

ECE 603

Fall, 2009

1)

(a)

Since  $A_1$  is countable,  $\exists f_1$  s.t.  $f_1: A_1 \rightarrow \mathbb{Z}_+$  is 1-to-1.  
Since  $A_2$  is countable,  $\exists f_2$  s.t.  $f_2: A_2 \rightarrow \mathbb{Z}_+$  is 1-to-1.

Now, here is my list:

$$f_1^{-1}(1)$$

$$f_2^{-1}(1)$$

$$f_1^{-1}(2)$$

$$f_2^{-1}(2)$$

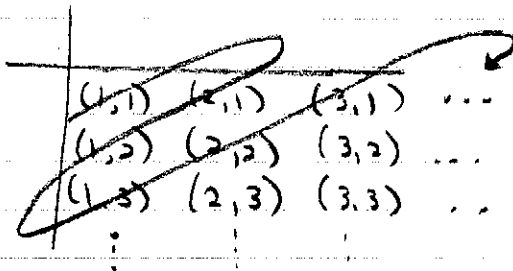
⋮

There may be repeats but this just means  $A_1 \cup A_2$  is at most countable  $\Rightarrow$  countable

(b)

Since  $A_n$  is countable,  $\exists f_n$  s.t.  $f_n: A_n \rightarrow \mathbb{Z}_+$  is 1-to-1.

Now consider



When I hit  $(i,j)$ , put  $f_i^{-1}(j)$  in the list

List:

$$f_1^{-1}(1)$$

$$f_2^{-1}(1)$$

$$f_1^{-1}(2)$$

$$f_1^{-1}(3)$$

⋮

} once again, repeats just make list smaller  $\Rightarrow$  no more than countable  $\Rightarrow$  countable.

2)

(a) Each such function can be expressed by a table:

$x$	$f(x)$
0	$f(0)$
1	$f(1)$

or an ordered pair  $(f(0), f(1))$

$\Rightarrow$  1-to-1 with  $\mathbb{N}_+ \times \mathbb{N}_+ \Rightarrow$  countably infinite

(b)

Similarly to (a), the function  $f$  can be specified by an ordered  $n$ -tuple:

$(f(1), f(2), \dots, f(n))$

$\Rightarrow$  1-to-1 with  $\mathbb{N}_+^n \stackrel{\Delta}{=} \underbrace{\mathbb{N}_+ \times \mathbb{N}_+ \times \mathbb{N}_+ \times \dots \times \mathbb{N}_+}_{n \text{ times}}$

But  $\mathbb{N}_+^n$  is countably infinite:

Obviously true for  $k=1$ . Now, assume it is true that  $\mathbb{N}_+^k$  is countable. Then,

$$\mathbb{N}_+^{k+1} = \mathbb{N}_+^k \times \mathbb{N}_+^1$$

But, if  $\mathbb{N}_+^k$  is countable,  $\mathbb{N}_+^k$  is 1-to-1 with  $\mathbb{N}_+$  and thus

$\mathbb{N}_+^{k+1}$  is 1-to-1 with  $\mathbb{N}_+ \times \mathbb{N}_+ \Rightarrow$  countable

Hence,  $\mathbb{N}_+^n$  is countable, for all  $n$ , by induction.

(c) A countable union of countable sets  $\Rightarrow$  countably infinite

(d)

Recall that any number in  $[0,1)$  (an uncountable set) can be written as:

$$0.a_1 a_2 a_3 a_4 \dots \quad a_i \in \{0,1\}$$

Now, for each function  $f: \mathbb{N}_+ \rightarrow \{0,1\}$ , think of its mapping:



and associate it with the number in  $[0,1)$  s.t.

$a_i = f(i)$ . Since this mapping is 1-1 (and  $[0,1)$  is uncountable),  $D$  is uncountable.

(e) Uncountable

$D \subseteq E$  and  $D$  uncountable.

(f) Finite

Think of a function:

$x$	$f(x)$	}	$\in \{0,1\}^{\mathbb{N}}$
1	$f(1)$		
2	$f(2)$		
3	$\vdots$		
$\vdots$	$\vdots$		
$n-1$	$f(n-1)$		
$n-2$	0		
$\vdots$	$\vdots$		

$|\{0,1\}^{\mathbb{N}}| = 2^{\mathbb{N}}$

(g)

Countably Infinite

Similar to (f), size is  $|\mathbb{Z}_+^{N-1}|$ .

(h)

Uncountable

Similar to (f), size is  $|\mathbb{R}^{N-1}|$ .

(i) Countably Infinite

There are fewer subsets (unordered) than ordered pairs:

$$I \subset \mathbb{Z}_+ \times \mathbb{Z}_+$$

(j)

Extend (i) to  $I_n$ : all  $n$ -element subsets of  $\mathbb{Z}_+$ . Each  $I_n$  is countable, and

$$J = \bigcup_{n=1}^{\infty} I_n$$

$\Rightarrow$  countably infinite

Note: This is different than all subsets of  $\mathbb{Z}_+$ , which is clearly uncountable (very similar to D)