Homework #1 Solutions

ECE 603 Fall, 2009

1)

(a)

Since A_1 is countable, $\exists f_1$ s.th. $f_1 : A_1 \to \mathbb{Z}_+$ is 1-to-1.
Since A_2 is countable, $\exists f_2$ s.th. $f_2 : A_2 \to \mathbb{Z}_+$ is 1-to-1

Now, here is my list:

$f_1^{-1}(1)$
$f_2^{-1}(1)$
$f_1^{-1}(2)$
$f_2^{-1}(2)$
$\vdots$

There may be repeats but this just means $A_1 \cup A_2$ is at most countable $\Rightarrow$ countable

(b)

Since A_n is countable, $\exists f_n$ s.th. $f_n : A_n \to \mathbb{Z}_+$ is 1-to-1.

Now consider

$$\begin{array}{cccc} (1,1) & (2,1) & (3,1) & \cdots \\ (1,2) & (2,2) & (3,2) & \cdots \\ (1,3) & (2,3) & (3,3) & \cdots \\ \vdots & \vdots & \vdots & \end{array}$$

When I hit (i,j), put $f_i^{-1}(j)$ in the list

List:

$f_1^{-1}(1)$
$f_2^{-1}(1)$
$f_1^{-1}(2)$
$f_1^{-1}(3)$
$\vdots$

$\Bigg\}$ once again, repeats just make list smaller $\Rightarrow$ no more than countable $\Rightarrow$ countable.

2)

(a) Each such function can be expressed by a table:

x	$f(x)$
0	$f(0)$
1	$f(1)$

or an ordered pair $(f(0), f(1))$

$\Rightarrow$ 1-to-1 with $\mathbb{Z}_+ \times \mathbb{Z}_+ \Rightarrow$ <u>countably</u> <u>infinite</u>

(b)

Similarly to (a), the function f can be specified by an ordered n-tuple:

$$(f(1), f(2), \ldots, f(n))$$

$$\Rightarrow \text{1-to-1 with } \mathbb{Z}_+^n \triangleq \underbrace{\mathbb{Z}_+ \times \mathbb{Z}_+ \times \mathbb{Z}_+ \times \cdots \times \mathbb{Z}_+}_{n \text{ times}}$$

But $\mathbb{Z}_+^n$ is <u>countably</u> <u>infinite</u>:

Obviously true for $k=1$. Now, assume it is true that $\mathbb{Z}_+^k$ is countable. Then,

$$\mathbb{Z}_+^{k+1} = \mathbb{Z}_+^k \times \mathbb{Z}_+^1$$

But, if $\mathbb{Z}_+^k$ is countable, $\mathbb{Z}_+^k$ is 1-to-1 with $\mathbb{Z}_+$ and thus

$\mathbb{Z}_+^{k+1}$ is 1-to-1 with $\mathbb{Z}_+ \times \mathbb{Z}_+ \Rightarrow$ countable

Hence, $\mathbb{Z}_+^n$ is countable, for all n, by induction.

(c) A countable union of countable sets $\Rightarrow$ <u>countably</u> <u>infinite</u>

(d)

Recall that any number in $[0,1)$ (an uncountable set) can be written as:

$$0.a_1 a_2 a_3 a_4 \ldots \qquad a_i \in \{0,1\}$$

Now, for each function $f: \mathbb{Z}_+ \to \{0,1\}$, think of its mapping:

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ f(1) & f(2) & f(3) & f(4) & f(5) \end{array}$$

and associate it with the number in $[0,1)$ s.th. $a_i = f(i)$. Since this mapping is 1-1 (and $[0,1)$ is uncountable), D is <u>uncountable</u>.

(e) <u>Uncountable</u>

$D \subseteq E$ and D uncountable

(f) <u>Finite</u>

Think of a function:

$$\begin{array}{cc} \underline{x} & \underline{f(\cdot)} \\ 1 & f(1) \\ 2 & f(2) \\ 3 & \vdots \\ \vdots & \\ N-1 & f(N-1) \\ N-2 & 0 \\ \vdots & \vdots \end{array} \left.\begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\right\} \in \{0,1\}^N$$

$$|\{0,1\}^N| = 2^N$$

(g) <u>Countably Infinite</u>

Similar to (f), size is $|\mathbb{Z}_+^{N-1}|$.

(h) <u>Uncountable</u>

Similar to (f), size is $|\mathbb{R}^{N-1}|$.

(i) <u>Countably Infinite</u>

There are fewer subsets (unordered) than ordered pairs:

$$I \subset \mathbb{Z}_+ \times \mathbb{Z}_+$$

(j) Extend (i) to I_n: all n-element subsets of $\mathbb{Z}^+$. Each I_n is countable, and

$$J = \bigcup_{n=1}^{\infty} I_n$$

$\Rightarrow$ <u>countably infinite</u>

<u>Note</u>: This is different than all subsets of $\mathbb{Z}_+$, which is clearly uncountable (very similar to D)