ECE 313 - Signals and Systems, Fall 2012

Practice Problems - Midterm Exam #1

Note: Keep checking the WWW site. More and more will be added over the weekend.

1. In class, we showed that an LTI system that is a differentiator turns $A_1 \cos(2\pi f_0 t)$ into $(-A_2 \sin(2\pi f_0 t))$, where A_1 and A_2 are (different) positive constants. (In application, we really do not care about the constants). But the clever student will note that you can also turn $\cos(2\pi f_0 t)$ into $(-\sin(2\pi f_0 t))$ using a delay, as follows.

   - Find the delay t_0 such that the system with impulse response $h(t) = \delta(t - t_0)$ with input $x(t) = \cos(2\pi f_0 t)$ has output $y(t) = -\sin(2\pi f_0 t)$. (No Fourier required for this part (gasp!)).
   - Find $H(f)$ for your answer in (a). Is it conjugate symmetric?
   - Find and plot the magnitude squared $|H(f)|^2$ and phase of $H(f)$.
   - Find $X(f)$ and use it, plus your answer to (b), to find $Y(f) = H(f)X(f)$. Take the inverse transform and show that you get $y(t) = -\sin(2\pi f_0 t)$.
   - [**This part is not a representative exam question.**] From an engineering perspective, how does the resulting system using a delay compare to the differentiator in terms of robustness to exact knowledge of f_0? For example, suppose you design for $f_0 = 1$ GHz, but the true input frequency is only known to within $f_0 \pm 0.1\%$.

2. It is very important to know how to use linearity and time-invariance in the following way: Given the ouput of a system for one or more inputs, use linearity and time-invariance to find the output to a more complicated input. Answer the following parts:

   - Consider an LTI system for which the response to (i.e. the output resulting from) the input $u(t)$ is some function $g(t)$. Find the response of the system to rect(t) in terms of $g(t)$.
   - Suppose that I know the response of an LTI system to rect(t) is $g_1(t)$, and I want the response to rect$(t/2)$. Consider two different techniques of solution, **only one of which is correct:**
     (a) I recognize that rect$(t/2)$ is just a scaling and thus simply scale the output to yield the response to rect$(t/2)$ as $g_1(t/2)$.
     (b) I recognize that rect$(t/2) =$ rect$(t - 1/2) +$ rect$(t + 1/2)$, and thus write that the response to rect$(t/2)$ will be $g_1(t - 1/2) + g_1(t + 1/2)$.

     Which technique is correct?
   - Consider an LTI system for which you know the response to rect(t/10) is $g_2(t)$ and the response to $\Lambda(t/10)$ is $g_3(t)$. Suppose you input the UMass function $m(t)$ from Homework 1 into the system. What is the output in terms of $g_2(t)$ and $g_3(t)$?

3. The following simple statement (which you should know well) is much more powerful than it might at first seem: "If a system with input $x(t)$ and output $y(t)$ is linear and time-invariant (LTI), then $Y(f) = H(f)X(f)$, where $Y(f) = \mathcal{F}(y(t))$, $X(f) = \mathcal{F}(x(t))$, and $H(f)$ is the Fourier transform of the impulse response $h(t)$." **Use this seemingly simple fact** to answer the following important questions:

   - For input $x(t) = 2\text{sinc}(4t)$, an LTI system has output $\text{sinc}^2(2t)$. Find $h(t)$ for this LTI system. (*Why is this important? For an LTI system, we know that the impulse response $h(t)$ allows us to find the output for any input; in other words, $h(t)$ completely characterizes the system.*)

- Suppose, for input $x(t) = 2\text{sinc}(4t)$, your boss asks you to design a system that has output $y(t) = \text{sinc}^2(4t)$. Can this be an LTI system? (*Hint: Try to find $H(f)$ and then $h(t)$*? This is an important engineering result: an LTI system cannot expand the bandwidth of a signal; thus, if you need to expand bandwidth, you will need to employ a non-LTI system.)
- Suppose an LTI system has input $x(t)$ and output $y(t)$. If I instead input $dx(t)/dt$, is the output $dy(t)/dt$?
- Following on the previous part, show the following important result: if I apply two LTI systems with impulse responses $h_1(t)$ and $h_2(t)$ successively to an input $x(t)$ to arrive at an output $y(t)$, the order in which those two systems is applied does not matter. In other words, two LTI systems are *commutative.*
- Since I cannot really input $\delta(\cdot)$ functions into a system, suppose that I always input $u(t)$ (the unit step function) and get the unit-step response $y_u(t)$. Using such, can I get the output for any input $x(t)$?
- Following on the previous part: it seems that, for an LTI system, given a single input $x(t)$ and its corresponding output $y(t)$, you can always find $h(t)$. Find an $x(t)$ such that this is not true.

4. An LTI system has impulse response $h(t) = e^{-5t}u(t)$. The signal $x(t) = 4\cos(2\pi 6t)$ is insert to the system and the output is $y(t)$. Find the power in the output $y(t)$.

5. We know that the Fourier transform of a real $x(t)$ is conjugate symmetric. But this is not very useful if you cannot validate it. Let $X(f) = \text{sinc}^2(6f)e^{-j2\pi 4f}$.

   - Find $X^*(-f)$.
   - Find $x(t)$. Is it real? Is this consistent with your answer to (a)?

6. The modulation property, which says the Fourier transform of $x(t)y(t)$ is $X(f) * Y(f)$, seems strange at first, because it sounds unlikely that you would ever want to turn multiplication into convolution. But there are surprisingly many applications.

   - One application is when the resulting convolution is simple, as in when the multiply in the time domain leads to a convolution with a $\delta(\cdot)$ function in the frequency domain. But that presumes you can convolve with $\delta(\cdot)$ functions. Find the following convolutions:
     (a) $\Lambda(f/4) * \delta(f-5)$. Is the corresponding time domain signal real? (Note: You should not have to do the inverse transform to figure out whether the signal is real.)
     (b) $(\Lambda(f/4) + \Lambda((f-10)/5)) * (\delta(f-5) + \delta(f-20))$
   - Another application is often when you have a system that is not linear and time-invariant (LTI). For example, suppose you have a "square-law" device that takes in $x(t)$ and outputs $y(t) = x^2(t)$.
     (a) Suppose $x(t) = \text{sinc}(4t)$. Recognizing that $x^2(t) = x(t) \cdot x(t)$, find the Fourier transform $Y(f)$ of the output $y(t)$.
     (b) Recall that the "bandwidth" of a signal $x(t)$ is the amount of the positive frequency axis that its Fourier transform covers. Find the bandwidths of $x(t)$ and $y(t)$ in the previous part.
     (c) Use the previous part to establish that the square-law device is not LTI.
     (d) Suppose a signal $x(t)$ of bandwidth W is input to this square-law device. What is the maximum bandwidth at the output? Can you think of an $x(t)$ for which there is no expansion of bandwidth?