1. (a) Suppose that a discrete-time sequence $x[n]$ is given by:

$$
\begin{align*}
    x[-2] &= 2 \\
    x[-1] &= 0 \\
    x[0] &= -2 \\
    x[1] &= -3 \\
    x[2] &= 4 \\
    x[3] &= 1 \\
    x[4] &= 1 \\
\end{align*}
$$

and $x[n] = 0$, otherwise. Without evaluating the full transform (which is unnecessary), use the definition and properties of the DTFT to find:

- $X(0)$
- $X(\pi)$
- $\int_{-\pi}^{\pi} X(\Omega) d\Omega$
- $\int_{-\pi}^{\pi} |X(\Omega)|^2 d\Omega$
- $\int_{-\pi}^{\pi} \left| \frac{dX(\Omega)}{d\Omega} \right|^2 d\Omega$

(b) Suppose that you have a discrete-time linear time-invariant (LTI) system for which somebody has given you the step-response; that is, they have given you the system response $g[n]$ to the input:

$$
u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$

How would you find the impulse response $h[n]$ of the LTI system? **Hint: Do this in the time domain. Think about how to exploit the linearity and time-invariance to obtain such.**

2. We know that a linear time-invariant (LTI) system is completely characterized by its impulse response $h[n]$. Suppose that the impulse response of our software module “FILTER” is given by:

$$h[n] = 0.5\delta[n] + \delta[n - 1] + 0.5\delta[n - 2]$$

where $\delta[n]$ is the standard discrete impulse function.

(a) Is this system causal? Is it stable?

(b) To test our code, we first give input $x_h[n] = \text{rect}(n/5)$ to the module “FILTER” and obtain the output $y_h[n]$. Find the output $y_h[n]$.

(c) Find the frequency response $H_{DT}(\Omega)$ of “FILTER” and plot its magnitude and phase.

(d) Suppose I put the signal $x[n] = \cos\left(\frac{2\pi}{3}n\right)$ into the filter. Find the output $y[n]$. 
3. Use trigonometric identities, Euler’s formula, and the shifting property of the discrete-time Fourier transform (DTFT) to find the sequence $x_1[n]$ whose Fourier transform is given by:

$$X_1(\Omega) = 1 + 2 \cos \Omega + 3 \cos^2 \Omega$$

4. Consider the following two sequences:

$$x_3[0] = 1$$
$$x_3[1] = 1$$
$$x_3[2] = 2$$
$$x_3[3] = 2$$

and $x_3[n] = 0$, otherwise. And

$$x_4[0] = 1$$
$$x_4[1] = 1$$
$$x_4[2] = 2$$
$$x_4[3] = 2$$
$$x_4[4] = 1$$
$$x_4[5] = 1$$
$$x_4[6] = 2$$
$$x_4[7] = 2$$

and $x_4[n] = 0$, otherwise. Suppose that we know $X_3(\Omega)$, the DTFT of $x_3[n]$. Find $X_4(\Omega)$, the DTFT of $x_4[n]$, in terms of $X_3(\Omega)$.

5. Note: This problem is not as hard as you might think at first.

Let $x[n]$ be a purely real sequence. You work in the laboratory to obtain the following information about $x[n]$:

(i) $x[n]$ is a causal sequence.

(ii) If $v[n] = x[n + 2]$, the Fourier transform $V(\Omega)$ of the discrete-time sequence $v[n]$ is purely real.

(iii) $$\frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\Omega)|^2 d\Omega = 16$$

(iv) $$x[0] = 2$$
6. A discrete-time linear and time-invariant (LTI) system is described by:

\[ y[n] = \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = x[n] \]

where \( x[n] \) and \( y[n] \) are the input and output of the system, respectively.

(a) Determine the frequency response \( H(\Omega) \) of the system.

(b) Find the impulse response \( h[n] \) of the system.

(c) Find \( y[n] \) if \( x[n] = 0.5^n u[n] \).

\( x[2] > 0. \)

Find the discrete-time sequence \( x[n] \).