ECE 564/645 - Digital Communications (Spring 2013)

Practice Problems #1

1. Consider the 2-dimensional vector channel $\underline{r} = \underline{s} + \underline{n}$ where $\underline{r}$ is the received vector, $\underline{s} = \underline{s}_i$ if message m_i is to be sent on $(0, T_s)$, and $\underline{n} = (n_1, n_2)^T$. Let n_1 and n_2 be independent Gaussian random variables with mean 0 and variance $\frac{N_0}{2}$ (i.e. this is an AWGN vector channel). Suppose there are eight possible **equally likely** messages ($M = 8$) and

$$\underline{s}_1 = (d, 0)^T \quad \underline{s}_2 = (2d, 0)^T$$

$$\underline{s}_3 = (-d, 0)^T \quad \underline{s}_4 = (-2d, 0)^T$$

$$\underline{s}_5 = (0, d)^T \quad \underline{s}_6 = (0, 2d)^T$$

$$\underline{s}_7 = (0, -d)^T \quad \underline{s}_8 = (0, -2d)^T$$

(a) Draw the signal space and decision regions for the MAP receiver.

(b) In terms of d and N_0:

- Find the Union Bound to the conditional symbol error probability $P(E|\underline{s}_1)$, and find the Union Bound to the conditional symbol error probability $P(E|\underline{s}_2)$.
- Remove as many summands as possible from the Union Bounds for $P(E|s_1)$ and $P(E|s_2)$, while still guaranteeing an upper bound is preserved in each case.

(c) Redo the second half of (b) in terms of E_s and N_0, where E_s is the average energy per signal.

(d) Without changing the energy of any of the signals, change the signal set to one whose MAP receiver has a smaller symbol error probability $P(E)$ **at large** $\frac{E_s}{N_0}$. An approximate sketch is sufficient, but be sure to justify your answer.

2. Consider the 1-dimensional vector channel $r = s + n$ where r is the received vector, $s = s_1 = -\frac{1}{2}$ if m_1 is sent and $s = s_2 = \frac{1}{2}$ if m_2 is sent, and n is a random variable with probability density function

$$p_n(x) \quad = \quad \begin{cases} |x| & |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Message m_1 is sent with probability p and message m_2 is sent with probability $1 - p$.

(a) Find the MAP decision rule as a function of p and draw the decision regions in r-space where each signal is chosen.

(b) Find the symbol error probability $P(E)$ as a function of p.

3. Consider the 2-dimensional vector channel $\underline{r} = \underline{s} + \underline{n}$ where $\underline{r}$ is the received vector, $\underline{s} = \underline{s}_i$ if message m_i is to be sent on $(0, T_s)$, and $\underline{n} = (n_1, n_2)^T$. Let n_1 and n_2 be independent Gaussian random variables with mean 0 and variance $\frac{N_0}{2}$ (i.e. this is an AWGN vector channel). Suppose there are four possible messages ($M = 4$) and

$$\underline{s}_1 = (0,0)^T$$

$$\underline{s}_2 = \left(\frac{d}{\sqrt{2}}, \frac{d}{\sqrt{2}}\right)^T$$

$$\underline{s}_3 = \left(0, \sqrt{2}d\right)^T$$

$$\underline{s}_4 = \left(-\frac{d}{\sqrt{2}}, \frac{d}{\sqrt{2}}\right)^T$$

(a) Assuming equally likely messages (i.e. $p(m_i) = \frac{1}{4}, \forall i$) and MAP reception, draw the decision regions in $\underline{r}$-space ($\underline{r} = (r_1, r_2)^T$), showing where each signal is chosen. You do not have to give precise intercepts and such for the boundaries - just approximate the picture.

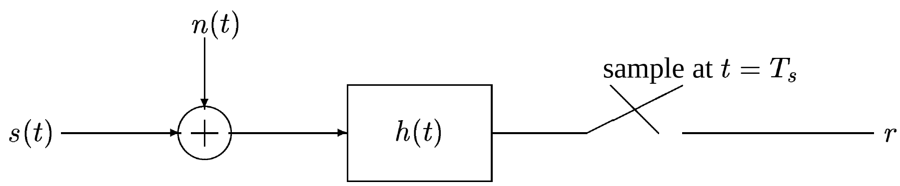
(b) For equally likely messages and the MAP receiver of (a):

- Find the symbol error probability $P(E)$ in terms of d and N_0.
- Find the symbol error probability $P(E)$ in terms of N_0 and the average symbol energy E_s.

(c) Now suppose $p(m_1) = \frac{1}{8}$, $p(m_2) = \frac{1}{8}$, $p(m_3) = \frac{1}{2}$, and $p(m_4) = \frac{1}{4}$ and assume the **same receiver** (no longer optimal) from (a) is employed.

- Find the symbol error probability $P(E)$ in terms of d and N_0.
- Find the symbol error probability $P(E)$ in terms of N_0 and the average symbol energy E_s.
- So far we have assumed that message m_i implies that signal $\underline{s}_i$ is sent. However, when the messages are not equally likely, it may be better to send (for example) $\underline{s}_1$ to represent m_4, $\underline{s}_2$ to represent m_3, $\underline{s}_3$ to represent m_2, and $\underline{s}_4$ to represent m_1. Specify the mapping from messages $\{m_i\}$ to signals $\{\underline{s}_i\}$ that minimizes $P(E)$ as a function of N_0 and the average symbol energy E_s. You only need to give a short sentence of justification for your answer.

4. Consider the following communication system:



where $n(t)$ is white Gaussian noise with power spectral density $\frac{N_0}{2}$, and $h(t)$ is a linear time-invariant filter applied to $r(t)$. Note that r is a scalar. For $t \in (0, T_s)$, we wish to communicate one of two **equally likely** messages m_1 and m_2 with respective signals

$$s_1(t) = 0$$

$$s_2(t) = \begin{cases} \sqrt{\frac{1}{T_s}}, & t \in (0, T_s) \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the optimal filter $h(t)$ and processing of r to implement the MAP receiver on $r(t)$ to decide between m_1 and m_2.

(b) Suppose instead of our filter from (a) that the factory ships us the filter

$$h(t) = \begin{cases} -\frac{t}{T_s}, & t \in (0, T_s) \\ 0, & \text{otherwise} \end{cases}$$

Given that we employ this filter in our system, what is the processing of r that minimizes the probability that the wrong message is chosen. Be sure to justify your answer.

5. Consider an independent and identically distributed (IID) discrete source $\{\hat{X}_k\}$ that can assume one of $M = 6$ values, where each source symbol has the following probability mass function:

$$p_{\hat{X}_k}(l) = \begin{cases} 0.4 & l = 0 \\ 0.2 & l = 1 \\ 0.2 & l = 2 \\ 0.1 & l = 3 \\ 0.06 & l = 4 \\ 0.04 & l = 5 \\ 0 & \text{otherwise} \end{cases}$$

(a) Suppose that the source symbols are losslessly encoded one at a time ($N = 1$). Find two (different) Huffman codes with distinct length distributions. In other words, the set $\{n(\underline{B}_i) : i = 1, \ldots, M\}$ for one of your Huffman codes must be different than it is for the other code, where $n(\underline{B}_i)$ is the length of the codeword assigned to the block $\underline{B}_i$. Give the rate of each of of your Huffman codes.

(b) You desire to know the signal-to-noise ratio $\frac{E_s}{N_0}$ of an AWGN channel, on which your competitor is using his BPSK communication system. Although your competitor knows how well his system works, he is generating proof of his system performance by collecting data in the following manner:

(a) He generates an independent and identically distributed sequence of bits, each equally likely to be 0 or 1.

(b) He sends the bits one at a time using his BPSK system over the AWGN channel of interest and looks at the demodulator output. If the output is correct, he records a 0 on his **notepad**. If the output is in error, he records a 1 on his **notepad**.

(c) For storage, he compresses the string of bits on his **notepad** into a **file**, using a Huffman coder on blocks of 10,000 bits. Note that since *he* knows the probability of error of his BPSK system on this AWGN channel, he can design such a Huffman code.

Fortunately, you are aware of the procedure that your competitor is using. You also are able to obtain the following information: After he has sent 20,000,000 bits over his system (and thus recorded 20,000,000 bits on his notepad), the size of the **file** is 8,000,000 bits. **Estimate** the signal-to-noise ratio of the AWGN channel.