ECE 745 - Advanced Communication Theory, Spring 2007

Midterm Exam

Monday, April 23rd, 6:00-9:00pm, ELAB 325

Overview

- The exam consists of five problems for 150 points. The points for each part of each problem are given in brackets - you should spend your **three hours** accordingly.
- The exam is closed book, but you are allowed **three page-side** of notes. Calculators are allowed for simple calculations (finding entropies, etc). I will provide all necessary blank paper.

Testmanship

- **Full credit will be given only to fully justified answers.**
- Giving the steps along the way to the answer will not only earn full credit but also maximize the partial credit should you stumble or get stuck. If you get stuck, attempt to neatly define your approach to the problem and why you are stuck.
- If part of a problem depends on a previous part that you are unable to solve, explain the method for doing the current part, and, if possible, give the answer in terms of the quantities of the previous part that you are unable to obtain.
- Start each problem on a new page. Not only will this facilitate grading but also make it easier for you to jump back and forth between problems.
- If you get to the end of the problem and realize that your answer must be wrong (e.g. the Fourier Transform of a real signal that is not conjugate symmetric), be sure to write “this must be wrong because . . . ” so that I will know you recognized such a fact.
- Academic dishonesty **will** be dealt with **harshly** - the *minimum penalty* will be an “F” for the course.

1. (a) An independent and identically distributed (IID) binary source has $P(X_k = 0) = 0.8$, $P(X_k = 1) = 0.2$, $\forall k$. Suppose that I take blocks of $N = 3$ bits at a time from this source as the input to my source coder.

   - [10] Find a Huffman code for this situation (i.e. a Huffman code that assigns to each 3-bit input sequence a variable-length output bit sequence.)
   - [10] Find the rate of your code from (a) in average output bits per input bit. **Also, (do not forget this part)**, use the entropy of the source to show that the rate of your Huffman code falls between some easily obtained (but non-trivial) bounds.
   - [10] Your boss has read a little information theory and has fallen in love with Shannon-Fano coding. Find a Shannon-Fano code for the source (i.e. a Shannon-Fano code that assign to each 3-bit input sequence a variable-length output bit sequence) and verify that its rate is consistent with your answers from (a) and (b). *Hint: The Shannon-Fano definition is non-constructive, but you should be able to find a code that meets the definition.*

   [10+EXTRA] (b) Now, your boss comes to you with two pieces of good news: (i) There are no complexity limits (i.e. you can let your N be as big as you want), and (ii) the source, which is stationary, is correlated. In fact, he tells you that, for any n, the conditional probability of the n^{th} symbol only depends on the previous symbol; that is,

$$p_{X_n|X_1,X_2,X_3,\ldots,X_{n-1}}(x_n|x_1,x_2,x_3,\ldots,x_{n-1}) = p_{X_n|X_{n-1}}(x_n|x_{n-1}) = \begin{cases} 0.9 & x_n = 0, x_{n-1} = 0 \\ 0.1 & x_n = 1, x_{n-1} = 0 \\ 0.4 & x_n = 0, x_{n-1} = 1 \\ 0.6 & x_n = 1, x_{n-1} = 1 \end{cases}$$

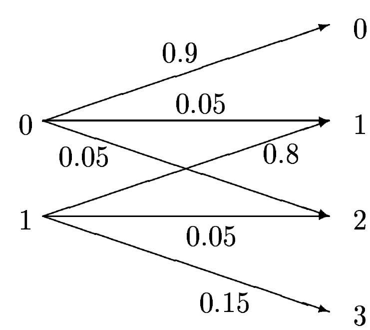
   Using stationarity and this conditional probability, you can show that the marginal probabilities are still $P(X_n = 0) = 0.8$, $P(X_n = 1) = 0.2$ (**extra credit** if you can show how one figures this out). For **regular credit**, find the rate that I tell my boss to expect out of the source coder **and compare it to your answer from (a).**

2. [30] Company WirelessX has mastered the ability to build circuits of any complexity, and thus they hire you because of your knowledge of Information Theory. The company desires to send an independent and identically distributed (IID) source, which generates symbols $X_1, X_2, X_3, \ldots$ at 50 symbols per second, and that is defined by the probability distribution:

$$P(X_k = \text{“}a\text{”}) = 0.5$$
$$P(X_k = \text{“}b\text{”}) = 0.2$$
$$P(X_k = \text{“}c\text{”}) = 0.3$$

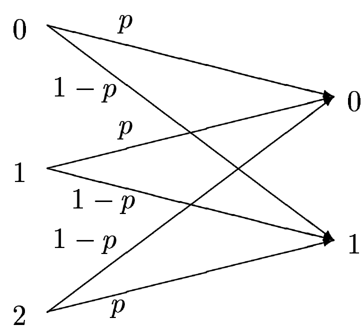
(a) You have access to a binary erasure channel (without feedback) with cross-over probability 0.1, which you recall from class has a capacity of 0.9 bits/channel use. Suppose that you are able to use this channel 100 times per second. Can this source be sent over this channel with error probability less than 10^{-10}? If not, explain why. If so, explain how one would go about doing this. Describe both the initial system construction (how codebooks are built, etc.) and also how an actual source message is transmitted through the system. *Obviously, you will not be able to find codebook lengths, etc, but be as clear and precise as possible.*

(b) You have access to the discrete memoryless channel (DMC) defined by:

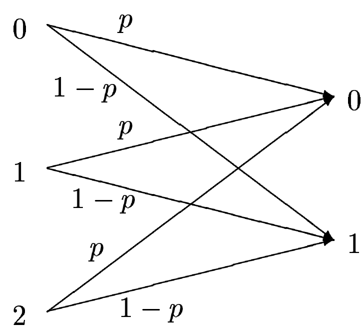


Suppose that you are able to use this channel 50 times per second. Can this source be sent over this channel with error probability less than 10^{-10}? If not, explain why. If so, explain how one would go about doing this. Describe both the initial system construction (how codebooks are built, etc.) and also how an actual source message is transmitted through the system. *Obviously, you will not be able to find codebook lengths, etc, but be as clear and precise as possible.*

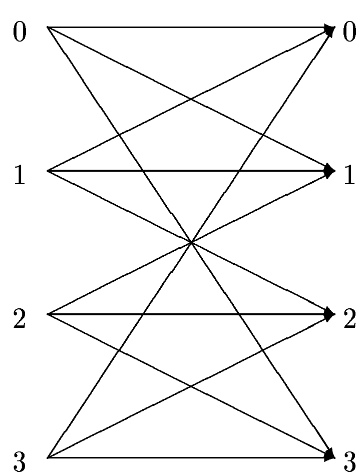
3. [10] (a) Find the capacity of the discrete memoryless channel (DMC) denoted by:



[10] (b) Find the capacity of the discrete memoryless channel (DMC) denoted by:



[10] (c) Find the capacity of the discrete memoryless channel (DMC) denoted by:



where each of the transitions shown has probability $\frac{1}{3}$.

4. This problem is about typical sequences. To make sure we are all using the same notation, define the typical set of length N and tolerance ϵ for an independent and identically distributed (IID) source sequence as:

$$A_\epsilon^{(N)} = \{\underline{x} : 2^{-N(H(X)+\epsilon)} \leq p_{\underline{X}}(x) \leq 2^{-N(H(X)-\epsilon)}\}$$

and here is the Typical Sequences Theorem that I want:

(a) $P(A_\epsilon^{(N)}) > 1 - \delta$ for N large enough.

(b) $|A_\epsilon^{(N)}| \leq 2^{N(H(X)+\epsilon)}$

(c) $|A_\epsilon^{(N)}| \geq (1-\delta)2^{N(H(X)-\epsilon)}$ for N large enough.

where $\epsilon > 0$ and $\delta > 0$ are arbitrary. *Note that I am* **purposely** *using the version that has both ϵ and δ.*

[10] (a) Suppose I have an independent and identically distributed (IID) binary source with $P(X_k = 0) = 0.8$ and $P(X_k = 1) = 0.2, \forall k$. I desire to design a fixed-length source code with error probability 10^{-10} that has a rate of less than 0.75 output bits per input bit. I employ a "typical set" approach. Pick the largest possible ϵ and δ (justify!). **Also (do not forget this part)**, explain why you would want to select the largest possible ϵ and δ, as opposed to other smaller ones that also would give the desired performance.

[10] (b) Suppose I have a source generating an independent sequence $\{X_k\}$, but the entries are not identically distributed. Instead, for odd k, $P(X_k = 0) = 0.8$, $P(X_k = 1) = 0.2$, and, for even k, $P(X_k = 0) = 0.9$, $P(X_k = 1) = 0.1$. Find the minimum rate of a fixed-length source code that is "almost lossless" (i.e. can achieve arbitrarily small error probability).

Notes (probably not needed, so do not let them confuse you):

(a) The source is not stationary (obviously).

(b) The source encoder and eventual decoder are "synchronized" (know that they are both starting at the odd index "k=1").

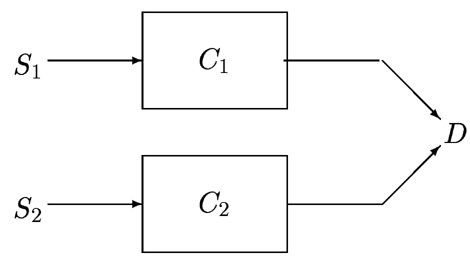
[10] (c) Suppose I have independent and identically distributed (IID) Binary Source 1 with $P(X_k = 0) = 0.6$ and $P(X_k = 1) = 0.4$, and I define its typical sets as $A_\epsilon^{(N)}$. Suppose I have independent and identically distributed (IID) Binary Source 2 with $P(Y_k = 0) = 0.56$ and $P(Y_k = 1) = 0.44$, and I define its typical sets as $B_\epsilon^{(N)}$. What can you say about the size of $A_\epsilon^{(N)} \bigcap B_\epsilon^{(N)}$ as a function of N and ϵ (only consider $0 < \epsilon < 0.1$)?

5. Before starting this problem, recall the following couple of lines for bounding the error probability during the achievability part of the derivation of the Channel Coding Theorem for discrete memoryless channels (DMCs):

$$\begin{aligned} P_e &< \frac{\delta}{2} + \sum_{i=1}^{2^{NR}-1} 2^{-N(I(X;Y)-3\epsilon)} \\ &< \frac{\delta}{2} + 2^{NR} 2^{-N(I(X;Y)-3\epsilon)}, \end{aligned}$$

where the first term upper bounds the probability that the received sequence is not jointly typical with the transmitted codeword, and the second term upper bounds the probability that the received sequence is jointly typical with another codeword. At the time, we were not concerned what happens when we got an error, but here we are interested in such. In particular, for $R > C$, we see that we have, on average, $2^{N(R-I(X;Y)+3\epsilon)}$ other codewords (other than the one transmitted) that are jointly typical with Y.

Now, suppose that we have the following situation. Two sources S_1 and S_2, with **non-interfering channels** with capacity C_1 and C_2, respectively, to a common destination D, desire to send a common message W (i.e. a message that they both know) of rate $R > C_1$ to the destination.



[20] Obviously, if $R < C_1 + C_2$, the senders can split the message to accomplish reliable communication. But, alas, Sender 1 insists on sending at rate $R > C_1$. Thus, it is your job as Sender 2 to come up with a scheme that transmits at the same time and uses **no** feedback from the destination (i.e. there is no way to know which "other codewords" are jointly typical with the output of channel 1 and are thus being confused with the correct one) to clear up the confusion. Give your scheme, prove it works, and indicate how big C_2 is required to be for your scheme. (*Hint: Think about how many bits you would need to send to appropriately reduce the second term in the error probability above, and how you might set up a scheme to accomplish such.*)