ECE 745 - Advanced Communication Theory, Spring 2002

Midterm Exam #1

March 27th, 6:00-8:00 p.m., ELAB 325

Overview

- The exam consists of five problems for 100 points. The points for each part of each problem are given in brackets - you should spend your **two hours** accordingly.
- The exam is closed book, but you are allowed **one page-side** of notes. Calculators are allowed, **but only** for doing simple addition, multiplication, taking logs, etc. (actually, you only need them to find a couple of entropies). I will provide all necessary blank paper.

Testmanship

- **Full credit will be given only to fully justified answers.**
- Giving the steps along the way to the answer will not only earn full credit but also maximize the partial credit should you stumble or get stuck. If you get stuck, attempt to neatly define your approach to the problem and why you are stuck.
- If part of a problem depends on a previous part that you are unable to solve, explain the method for doing the current part, and, if possible, give the answer in terms of the quantities of the previous part that you are unable to obtain.
- Start each problem on a new page. Not only will this facilitate grading but also make it easier for you to jump back and forth between problems.
- If you get to the end of the problem and realize that your answer must be wrong, be sure to write "this must be wrong because . . . " so that I will know you recognized such a fact.

- Academic dishonesty will be dealt with harshly - the *minimum penalty* will be an “F” for the course.

1. Suppose that I have a source that outputs a stationary random process $\ldots, X_{-1}, X_0, X_1, \ldots$, where each X_i is drawn from one of a finite number of values.

   [15] (a) Suppose each random variable is drawn from the alphabet $\mathcal{X} = \{0, 1, 2\}$, where the marginal probability mass function of X_i is given by:

   $$p_{X_i}(x) = \begin{cases} 0.5, & x = 0 \\ 0.3, & x = 1 \\ 0.2, & x = 2 \end{cases},$$

   and suppose that the X_i are mutually independent (thus (X_i) is an independent and identically distributed (IID) sequence).

   - Design a Huffman code that takes blocks of length 2 (i.e. $N = 2$) and find its rate (in bits/symbol).
   - Without finding the Huffman code (this would take far too much time!), find lower and upper bounds to the rate (in bits/symbol) of a Huffman code that takes symbols six at a time (i.e. $N = 6$). Full credit goes to the tightest bounds.

   [10] (b) Now suppose each sequence element is drawn from the binary alphabet $\mathcal{X} = \{0, 1\}$, where the marginal probability mass function of X_i is given by:

   $$p_{X_i}(x) = \begin{cases} 0.5, & x = 0 \\ 0.5, & x = 1 \end{cases}$$

   For this part, the random variables in the sequence are no longer mutually independent; however, we know that the source is Markovian, which means that the distribution of the next variable of the sequence depends only on the last value of the sequence; that is,

   $$P(X_{i+1} = y | X_i = x_i, X_{i-1} = x_{i-1}, X_{i-2} = x_{i-2}, \ldots) = P(X_{i+1} = y | X_i = x_i)$$

   for any $y, x_i, x_{i-1}, \ldots$ and the conditional distribution of a random variable in the sequence given the previous is given by:

   $$P(X_{i+1} = x | X_i = y) = \begin{cases} 0.75, & x = 0, y = 0 \\ 0.25, & x = 1, y = 0 \\ 0.25, & x = 0, y = 1 \\ 0.75, & x = 1, y = 1 \end{cases}$$

   Using only entropy-based arguments (no need to design any codes), find upper and lower bounds on the rate (in bits/symbol) of a Huffman code for such a source for two separate cases: (1) one that takes symbols six at a time ($N = 6$), (2) one that takes some very large number of symbols at a time (N very large).

2. A discrete-valued source outputs an independent and identically distributed (IID) sequence of random variables (X_i), each drawn from the alphabet $\mathcal{X} = \{a_1, a_2, \ldots, a_Q\}$. I have not been able to find the probability mass function $p_{X_i}(x)$; however, I have been able to show that for any $\epsilon > 0$, there are sets $B_1, B_2, B_3, \ldots$ that have the following three properties:

   (i) B_N is a subset of $\mathcal{X}^N$.

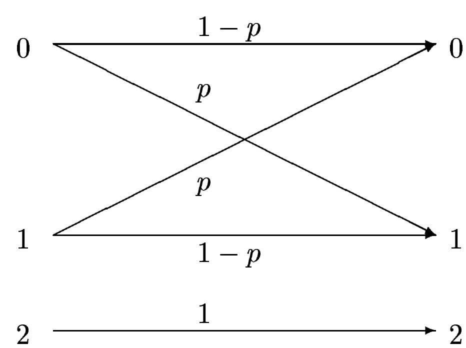
   (ii) $P((X_1, X_2, \ldots, X_N)^T \in B_N) \to 1$ as $N \to \infty$.

   (iii) $|B_N| = 4^{3\epsilon N + 5N}$.

   [10] (a) What can be said about the entropy of the source? (The right answer with a heuristic argument is okay for this part.)

   [5] (b) Make your answer to part (a) rigorous.

3. Consider the three-input, three-output discrete memoryless channel:



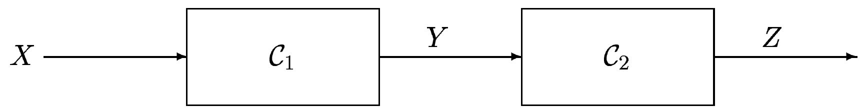
   Unfortunately, finding the capacity of this channel is not easy given the tools we have covered in class. Thus, consider the following special cases.

   [8] (a) Assume that $p = 0$. Find the capacity of the channel.

   [15] (b) Assume that $p = \frac{1}{2}$. Find the capacity of the channel **and give a simple encoding/decoding algorithm for achieving that capacity.**

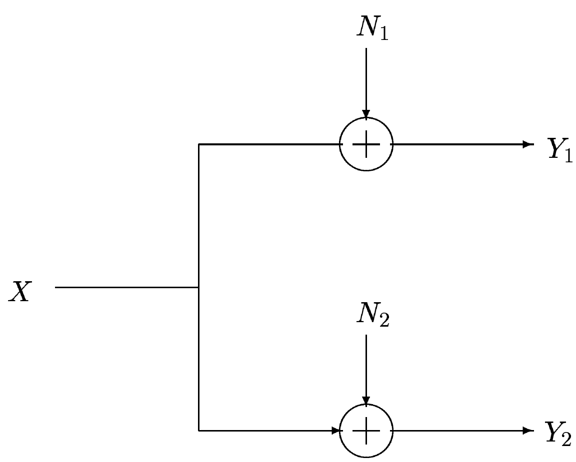
   [7] (c) For any p, show that $C \geq (\frac{3}{2} - \frac{1}{2}\mathcal{H}_2(p))$, where $\mathcal{H}_2(\cdot)$ is the binary entropy function.

4. [15] Consider two discrete memoryless channels $\mathcal{C}_1$ and $\mathcal{C}_2$ with capacities C_1 and C_2, respectively. Assume that the output alphabet of $\mathcal{C}_1$ is equal to the input alphabet of $\mathcal{C}_2$ and that they are connected as:



Show that the capacity of the cascade of channels (i.e. the channel with input X and output Z) is less than or equal to the minimum of C_1 and C_2.

5. [15] We desire to send separate messages to two separate users, but we only have a single input X with power constraint $E[X^2] \leq P$ to do so over a channel as shown below. One user observes Y_1 and the other user observes Y_2. The noise variables N_1 and N_2 are Gaussian with zero mean and variance σ_1^2 and σ_2^2, respectively, and the noise is independent for each use of the channel. Let R_i be the information rate (in bits of the i^{th} user's message per channel use) to the ith user.



Assume $\sigma_1^2 \leq \sigma_2^2$ (i.e. the channel for user 1 is as good as or better than the channel for user 2). Give an encoding and decoding method for achieving the rate pair (R_1, R_2), where

$$R_1 \approx \frac{1}{2}\log_2\left(1 + \frac{\gamma P}{\sigma_1^2}\right)$$
$$R_2 \approx \frac{1}{2}\log_2\left(1 + \frac{(1-\gamma)P}{(\gamma P + \sigma_2^2)}\right)$$

for any $0 \leq \gamma \leq 1$.