ECE 745 - Advanced Communication Theory, Spring 1999

Note (for 2007): This was an untimed exam!

Midterm Exam #1

April 1st, 6:00-8:00, ELAB 323

Overview

- The exam consists of six problems for 120 points. The points for each part of each problem are given in brackets - you should spend your **two hours** accordingly.
- The exam is closed book, but you are allowed **one page-side** of notes. Calculators are allowed, **but only** for doing simple addition, multiplication, taking logs, etc. (actually, you only need them to find a couple of entropies). I will provide all necessary blank paper.

Testmanship

- **Full credit will be given only to fully justified answers.**
- Giving the steps along the way to the answer will not only earn full credit but also maximize the partial credit should you stumble or get stuck. If you get stuck, attempt to neatly define your approach to the problem and why you are stuck.
- If part of a problem depends on a previous part that you are unable to solve, explain the method for doing the current part, and, if possible, give the answer in terms of the quantities of the previous part that you are unable to obtain.
- Start each problem on a new page. Not only will this facilitate grading but also make it easier for you to jump back and forth between problems.
- If you get to the end of the problem and realize that your answer must be wrong, be sure to write "this must be wrong because . . . " so that I will know you recognized such a fact.

- Academic dishonesty will be dealt with harshly - the *minimum penalty* will be an “F” for the course.

1. Huffman Coding:

   [15] (a) For a source producing an **IID sequence** of discrete random variables, each drawn from source alphabet $\mathcal{X}$, it has been found that a Huffman code on blocks of length 2 (i.e. $N = 2$ source symbols are taken at a time) has rate 2 bits/symbol, and a Huffman code on blocks of length 3 (i.e. $N = 3$ source symbols are taken at a time) has rate 1.6 bits/symbol.

   - Find upper and lower bounds to the **first-order entropy** of the source.
   - Find upper and lower bounds to the **entropy rate** of the source.
   - What can be deduced about the size of the source alphabet?

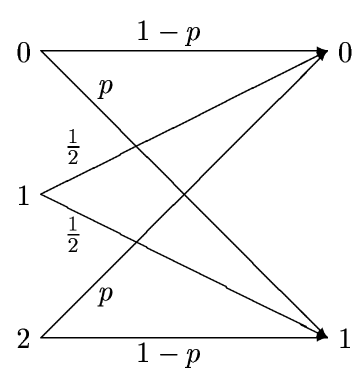
   [10] (b) Does there exist a source producing an **IID sequence** of discrete random variables, and integers $N \in \{1, 2, 3, 4, \ldots\}$ and $M \in \{2, 3, 4, \ldots\}$, such that a Huffman code on blocks of length N has (strictly) **smaller** rate (in bits/symbol) than a Huffman code on blocks of length MN?

2. [15] Let X, Y, and Z be **independent** discrete random variables, where $P(Z = 0) = \frac{1}{2}$ and $P(Z = 1) = \frac{1}{2}$. Define the random variable U as follows:

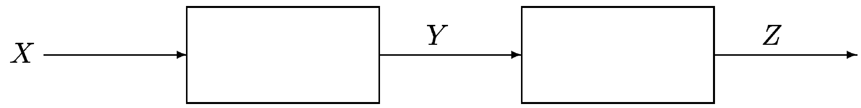
$$U = \begin{cases} X, & Z = 0. \\ Y, & Z = 1. \end{cases}$$

   Show $H(U) \geq \frac{1}{2}(H(X) + H(Y))$.

3. [15] Find the **capacity** of the channel shown below:



4. Consider discrete random variables X, Y, and Z. Suppose that X and Z are conditionally independent given Y as implied by the diagram below:



Are each of the following statements true or false? ("True" means for **all** X, Y, and Z such that X and Z are conditionally independent given Y. "False" means **there exists** X, Y, and Z, where X and Z are conditionally independent given Y, such that the statement does not hold). Be sure to give justification for your answers.

[5] (a) $I(X;Y|Z) = 0$

[5] (b) $H(X|Z) = H(X)$

[5] (c) $H(X|Y,Z) = H(X|Y)$

[5] (d) $I(X;Z) \leq H(Y)$

[5] (e) $H(Y|Z) \leq H(X|Z)$

[5] (f) $I(X;(Y,Z)) = I(X;Y)$

5. A source produces an **IID sequence** of discrete random variables, each drawn from source alphabet $\mathcal{X} = \{a, b, c, d\}$, where the elements have respective probabilities $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{8}$. We want to encode this source with a **fixed length almost lossless** source code. Assume that a (very large number) N source symbols are taken at a time and mapped into K binary digits.

   [5] (a) Assuming N can be chosen arbitrarily large, find the infimum of the rates $\frac{K}{N}$ for which the decoding error probability can be made arbitrarily small.

   [10] (b) Now suppose that I restrict each of the **output** sequences of K binary digits to have **at most** $\frac{K}{4}$ 1's. Assuming N can be chosen arbitrarily large, **estimate** the infimum of the rates $\frac{K}{N}$ for which the decoding error probability can be made arbitrarily small (Yes, there is a nice closed form solution).

6. **Note: You can do part (b) without doing part (a)!**

[10](a)Let X and Y be two random variables, each with finite second moment. Assume X is continuous. The minimum mean squared error estimate of X given $Y = y$ is given by:

$$\hat{X}_{MMSE}(y) = E[X|Y = y] \tag{1}$$

This estimator minimizes the mean squared error

$$\sigma_E^2 = E_{X,Y}[(X - \hat{X}(Y))^2]$$

over all estimators $\hat{X}(y)$. Use (1) to show that for any estimator, σ_E^2 is lower bounded as:

$$\sigma_E^2 \geq \frac{1}{2\pi e} 2^{2h(X|Y)}$$

where $h(X|Y)$ is the conditional differential entropy of X given Y.

[10] (b) Now suppose that X is a Gaussian random variable with mean 1 and variance σ_X^2 and that Y is a Q-level quantization of X as shown below (thus, $|\mathcal{Y}| = Q$). Use the result of part (a) to show that for the estimator $\hat{X}(y)$ in the block diagram below,

$$E_{X,Y}[(X - \hat{X}(Y))^2] \geq \frac{\sigma_X^2}{Q^2}$$