Midterm #2 Solutions

ECE 603 — Fall, 2009

1) (a)

$$P((0,1)) = 1 \Rightarrow c\cdot({}^{1}/_{4}) + c\cdot{}^{1}/_{2} = 1$$

$$3/4\, c = 1 \Rightarrow c = 4/3$$

(b)

$$P(|\omega - 1/2| > 1/4) = P(\{\omega < 1/4\} \cup \{\omega > 3/4\})$$

$$= P(\{\omega < 1/4\}) + P(\{\omega > 3/4\})$$

$$= P((0, 1/4)) + P((3/4, 1))$$

$$= 4/3 \cdot (1/4^2 - 0^2) + 4/3\,(1 - 3/4)$$

$$= 1/12 + 1/3 = 5/12$$

(c)

$$P(X=0) = P(\omega \in (0, 1/2)) = 4/3\,((1/2)^2 - 0^2) = 1/3$$

$$P(X=1) = P(\omega \in (1/2, 1)) = 4/3\,(1 - 1/2) = 2/3$$

sum to 1. ✓

$$f_X(x) = 1/3\,\delta(x) + 2/3\,\delta(x-1)$$

(d)

$$P(X=0, Y=0) = P(\omega \in (0, 1/4)) = 4/3\,((1/4)^2 - 0) = 1/12$$

$$P(X=0, Y=1) = P(\omega \in (1/4, 1/2)) = 4/3\,((1/2)^2 - (1/4)^2) = 1/4$$

$$P(X=1, Y=1) = P(\omega \in (1/2, 3/4)) = 4/3\,(3/4 - 1/2) = 1/3$$

$$P(X=1, Y=0) = P(\omega \in (3/4, 1)) = 4/3\,(1 - 3/4) = 1/3$$

$$f_{X,Y}(x,y) = 1/12\,\delta(x,y) + 1/4\,\delta(x, y-1) + 1/3\,\delta(x-1, y-1) + 1/3\,\delta(x-1, y)$$

(e)

$$E[XY] = 0\cdot 0\cdot 1/12 + 0\cdot 1\cdot 1/4 + 1\cdot 1\cdot 1/3 + 1\cdot 0\cdot 1/3 = 1/3$$

2)

(a) $P(X_1 = 1) = P(\omega = \text{Apple}) = 1/5$

$P(X_1 = 2) = P(\omega = \text{Banana}) = 1/5$

$P(X_1 = 3) = P(\omega = \text{Lime}) = 1/5$

$P(X_1 = 4) = P(\omega = \text{Pear}) = 1/5$

$P(X_1 = 5) = P(\omega = \text{Orange}) = 1/5$

$$f_{X_1}(x) = 1/5\,\delta(x) + 1/5\,\delta(x-1) + 1/5\,\delta(x-2) + 1/5\,\delta(x-3) + 1/5\,\delta(x-4)$$

(b)

$\{X_n\} \to 0$ in every way, which I can get by showing:

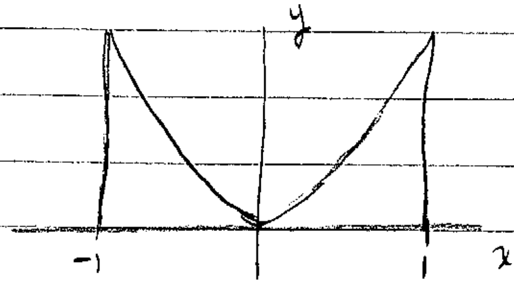
$\{X_n\} \xrightarrow{a.s.} 0$: For any ω, $X_n(\omega) \to 0$; thus,

$$P(\{\omega : X_n(\omega) \to 0\}) = 1$$

$\{X_n\} \xrightarrow{m.s.} 0$: $E[|X_n - X|^2] = E[|X_n|^2]$

$$\leq E[|5/n|^2] = 25/n^2 \to 0$$

3)



(a) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$

$\int_{-1}^{0}\int_{0}^{x^2} cx^2y\,dy\,dx + \int_{0}^{1}\int_{0}^{x^2} cx^2y\,dy\,dx$

$= \int_{-1}^{0} cx^2y^2/_2 \Big|_0^{x^2} dx + \int_{0}^{1} cx^2y^2/_2 \Big|_0^{x^2} dx$

$= \int_{-1}^{0} cx^6/_2\,dx + \int_{0}^{1} cx^6/_2\,dx$

$= c \cdot x^7/_{14} \Big|_{-1}^{0} + c \cdot x^7/_{14} \Big|_0^1 = c/_7 \Rightarrow c = 7$

(b) $p(x > 1/_2) = \int_{1/2}^{1}\int_{0}^{x^2} 7x^2y\,dy\,dx$

$= \int_{1/2}^{1} 7x^2y^2/_2 \Big|_0^{x^2} dx$

$= \int_{1/2}^{1} 7x^6/_2\,dx$

$= 1/_2\, x^7 \Big|_{1/2}^{1}$

$= 1/_2\left(1 - (1/_2)^7\right)$

$= \frac{1}{2} \cdot \frac{127}{128} = \frac{127}{256}$

(c)

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

$$= \int_{-1}^{-\sqrt{y}} 7x^2 y\,dx + \int_{\sqrt{y}}^{1} 7x^2 y\,dx$$

$$= 2 \cdot \tfrac{7}{3} x^3 y \Big|_{\sqrt{y}}^{1}$$

$$= 2 \cdot \tfrac{7}{3} y \left(1 - y^{3/2}\right)$$

$$= \tfrac{14}{3}\left(y - y^{5/2}\right)$$

integrates to 1! ✓

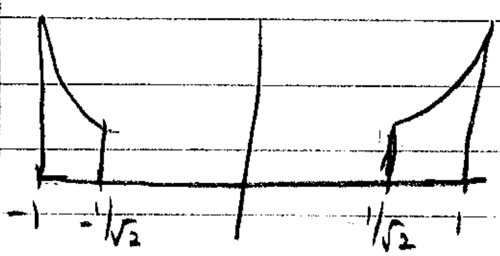
$$\Rightarrow f_Y(y) = \begin{cases} \tfrac{14}{3}\left(y - y^{5/2}\right) & 0 \le y \le 1 \\ 0, & \text{else} \end{cases}$$

$$f_{X|Y}(x|y) = \begin{cases} 3x^2 / 2\left(1 - y^{3/2}\right) & 0 \le y \le 1,\ \sqrt{y} \le |x| \le 1 \\ 0, & \text{else} \end{cases}$$

integrates to 1! ✓

(d)

$$f_{X|Y}(x|0.5) = \begin{cases} 3x^2 / 2\left(1 - (1/2)^{3/2}\right) & 1/\sqrt{2} \le |x| \le 1 \\ 0, & \text{else} \end{cases}$$



$|x| \approx 1$ are most likely.

4)

(a) $X_4 \sim N(3/4, 1/4)$

$$P(X_4 > 3/2) = 1 - P(X_4 \le 3/2)$$

$$\left[\frac{3/2 - 3/4}{1/2} = 3/2\right] \overset{\downarrow}{=} 1 - \left(\tfrac{1}{2} + \operatorname{erf}(3/2)\right)$$

$$= 1/2 - \operatorname{erf}(3/2)$$

(b)
$$P(X_4 > 3/2) = 1/2 - P(3/4 < X_4 < 3/2)$$

$$= 1/2 - \int_{3/4}^{3/2} \frac{1}{\sqrt{2\pi \cdot 1/4}} e^{-(x-3/4)^2/2\cdot 1/4}\,dx$$

$u = \dfrac{x - 3/4}{\sqrt{1/2}}$
$$\overset{\downarrow}{=} 1/2 - \int_0^{3/4\cdot\sqrt{2}} \frac{1}{\sqrt{2\pi\cdot 1/4}} e^{-u^2}\sqrt{1/2}\,du$$

$$= 1/2 - \frac{1}{\sqrt{\pi}} \int_0^{3/4\cdot\sqrt{2}} e^{-u^2}\,du$$

$$= 1/2 - 1/2\,\operatorname{Erf}\left(3/4\cdot\sqrt{2}\right)$$

This part removed from exam! Ignore.

Y is Gaussian: $E[Y] = E[6X_1 + 4] = 6E[X_1] + 4 = 4$

$$E[Y^2] = E[(6X_1+4)^2] = E[36X_1^2 + 48X_1 + 16]$$

$$= 36E[X_1^2] + \cancel{48E[X_1]}^{0} + 16$$

$$= 52$$

$$\Rightarrow \sigma_Y^2 = 52 - 4^2 = 36 \qquad f_Y(y) = \frac{1}{\sqrt{2\pi 36}} e^{-(y-4)^2/2\cdot 36}$$

(c) I claim $\{X_n\} \to X = 1$ in all three ways

$$E[|X_n - X|^2] = E[(X_n - 1)^2] = E[X_n^2] - 2E[X_n] + 1$$

$$= \operatorname{Var}(X_n) + [E[X_n]]^2 - 2E[X_n] + 1$$

$$= 1/n + (1 - 1/n)^2 - 2(1 - 1/n) + 1$$

$$= 1/n + \cancel{1} - \cancel{2/n} + 1/n^2 - \cancel{2} + \cancel{2/n} + \cancel{1} \to 0$$

$\{X_n\} \xrightarrow{m.s.} 1$

$\Downarrow$

$\{X_n\} \xrightarrow{P} 1 \Rightarrow \{X_n\} \xrightarrow{D} 1$

5)

(a) $W = aX + b$

$$E[W] = aE[X] + b = 3a + b = 0$$

$$\begin{aligned}Var[W] &= E[W^2] - (E[W])^2 \\ &= E[a^2X^2 + 2abX + b^2] - (3a+b)^2 \\ &= a^2 \cdot 18 + \cancel{6ab} \cancel{+b^2} - (9a^2 + \cancel{6ab} \cancel{+b^2}) \\ &= 9a^2 = 1\end{aligned}$$

$\Rightarrow a = 1/3$ and $b = -1$ <u>or</u> $a = -1/3$ and $b = 1$

(b) $E[(X+Y)^2] = E[X^2] + 2E[XY] + E[Y^2] = 6$

$$\Rightarrow E[X^2] + E[Y^2] = 3$$

$$\Rightarrow \sigma_X^2 + \sigma_Y^2 = 3$$

but you also know $\rho_{X,Y} \le 1$

$$\Rightarrow \frac{3/2}{\sigma_X \sigma_Y} \le 1$$

$$\Rightarrow 9/4 \le \sigma_X^2 \sigma_Y^2$$

which means σ_X^2 and σ_Y^2 cannot be too small:

$$9/4 \le \sigma_X^2 (3 - \sigma_X^2)$$

is only true if $\sigma_X^2 = 3/2$ (and thus $\sigma_Y^2 = 3/2$)

(take a derivative - right side maximized at $\sigma_X^2 = 3/2$.