ECE 564/645 - Digital Communication Systems (Spring 2013)

Midterm Exam #2

Tuesday, April 16th, 7:00-9:00pm, Marston 211

Overview

- The exam consists of four problems for 100 points. The points for each part of each problem are given in brackets - you should spend your **two hours** accordingly.
- The exam is closed book, but you are allowed **two page-sides** of notes. Calculators are **not** allowed. I will provide all necessary blank paper.

Testmanship

- **Full credit will be given only to fully justified answers.**
- Giving the steps along the way to the answer will not only earn full credit but also maximize the partial credit should you stumble or get stuck. If you get stuck, attempt to neatly define your approach to the problem and why you are stuck.
- If part of a problem depends on a previous part that you are unable to solve, explain the method for doing the current part, and, if possible, give the answer in terms of the quantities of the previous part that you are unable to obtain.
- Start each problem on a new page. Not only will this facilitate grading but also make it easier for you to jump back and forth between problems.
- If you get to the end of the problem and realize that your answer must be wrong (e.g. a negative probability), be sure to write "this must be wrong because ..." so that I will know you recognized such a fact.
- Academic dishonesty **will** be dealt with **harshly** - the *minimum penalty* will be an "F" for the course.

1. A stream of independent information bits that are equally likely to be 0 or 1 are grouped into pairs to form vectors $\underline{u}$ of length 2, which are used by the channel encoder to form codewords $\underline{u}^T G$, where

$$G = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

[15] (a) Each part is worth three points, and no justification is required.

- What is the rate r of this code ?
- Give the set of codewords $\mathcal{C}$.
- Give the weight enumerator of the code **and identify** d_{min}.
- Give the parity check matrix H of the code.
- How many syndromes are there ?

[10] (b) Does this code have the largest minimum distance of any $(6, 2)$ linear block code?:

- If you say **"Yes"**: Prove that the d_{min} of this code is the best of any $(6, 2)$ code.
- If you say **"No"**: Find a $(6, 2)$ code with a better minimum distance.

2. The coded bits out of the encoder of the code from **part (a) of Problem 1** are mapped in **pairs** by an amplitude-shift keying (ASK) modulator to the signals whose signal space equivalents s are as follows:

bits	s
00	3
01	1
11	-1
10	-3

The resulting sequence of ASK symbols is sent across a channel that is modeled as a 1-dimensional vector channel $r = s + n$, where n is a zero-mean Gaussian random variable with variance $\frac{N_0}{2}$.

Two observations for your aid: (1) There are three transmitted (and thus also three received) symbols for each codeword. (2) The transmitted ASK symbols are equally likely.

[10] (a) Assuming hard-decision decoding is done at the receiver (i.e. the demodulator forms its best estimate of each transmitted symbol and passes along the associated **bits** to the channel decoder, which decodes as if each bit is an independent output of a binary symmetric channel), give the information bits output by the channel decoder when the following received sequence is **input to the demodulator** (*Hint: Since the code is so small, standard array decoding may not be the simplest option for hard-decision decoding the code, so think carefully before building the standard array.*):

-1.0, -2.1, -2.1, 2.6, 1.2, 0.3

[10] (b) Assuming soft-decision decoding is done at the receiver (i.e the optimal processing of the received symbol sequence to find the transmitted codeword), give the information bits output by the channel decoder for the following received symbol sequence.

-1.0, -2.1, -2.1, 1.5, 1.9, 3.5

[10] (c) Find the Union Bound to $P(E|\underline{c}_0)$, the probability of a codeword error given the all zeroes codeword was sent, when soft-decision decoding is employed. The solution can be left in units of the vector space (i.e. there is no need to convert to the average symbol energy E_s). **If the overall codeword error probability is $P(E)$, does $P(E) = P(E|\underline{c}_0)$?**

3. [10] (a) Show there is **no** $(7, 2)$ linear block code that has minimum distance $d_{min} = 5$.

   (Note: Part (a) does **not** depend on the information below.)

   [10] (b) Given a code $\mathcal{C}$ with parity check matrix H, we can form other codes by deleting columns of the parity check matrix H. This is called *shortening* the code:

   - When a code is *shortened,*, what happens to its rate r? ("smaller", "same", "larger" are possible answers).
   - When a code is *shortened,*, what happens to its minimum distance d_{min}? ("smaller", "same", "larger" - or a combination of some of these choices - are possible answers).

   [15] (c) The columns of the parity check matrix of a binary $(31, 26)$ Hamming code consists of all of the non-zero binary 5-tuples.

   - What is the minimum distance of the $(31, 26)$ Hamming code?
   - Show that the $(31, 26)$ Hamming code is a perfect code; that is, show that it has precisely the right number of syndromes (and no extra ones) to correct a single error.
   - Find the two-error correcting (i.e. $d_{min} \geq 5$) code of highest rate that can be found by shortening the $(31, 26)$ Hamming code. (*Hint: Part (a) may be useful in justifying your answer.*).

4. [10] Consider the $(2, 1)$ constraint-length 3 convolutional code given in class, with state diagram given by:

   Suppose that we input four information information bits to the encoder, followed by two input zeros to drive the encoder to the "00" state. The resulting output sequence of 12 coded bits is sent across a binary symmetric channel (BSC) with crossover probability $p < \frac{1}{2}$ and the following sequence is received:

   010110011000

   Find the most likely information bits. Feel free to draw only a single trellis and erase branches in your trellis as needed, leaving only the optimal path (i.e. no reason to show a separate trellis at the end of every time stage).

5. [645 only] Suppose we are given an (n, k) linear block code $\mathcal{C}$ with generator matrix G. The *dual* code $\mathcal{C}^\perp$ is defined as the code with parity check matrix $\tilde{H} = G$ (i.e. the parity check matrix of the dual code is the generator matrix of the original).

   [5] (a) Give the length, number of information bits per codeword, and rate of the dual code $\mathcal{C}^\perp$ in terms of the parameters n and k of $\mathcal{C}$.

   [5] (b) Show that if a vector $\underline{x}$ is in $\mathcal{C}^\perp$, then $\underline{x} \oplus \underline{c}_i = 0$ for all $\underline{c}_i$ in $\mathcal{C}$.

   [5] (c) Does $\mathcal{C} \bigcup \mathcal{C}^\perp$ contain all vectors of length n ?

   [5] (d) Find the weight enumerator of the dual code to a (7.4) Hamming code. (*Recall that the columns of the parity check matrix of a binary* $(7, 4)$ *Hamming code consists of all of the non-zero binary* 3-*tuples - in any order you like.*)