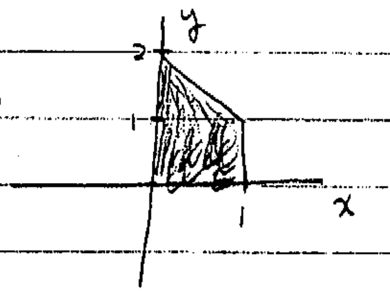
Midterm #2 Solutions

ECE 314 Spring 2012

1)

$$f_{X,Y}(x,y) = \begin{cases} kx, & 0 \le x \le 1,\ 0 \le y \le 2-x \\ 0, & \text{else} \end{cases}$$

(a)



(b) $\int_0^1 \int_0^{2-x} kx\,dy\,dx = \int_0^1 (kxy)\Big|_0^{2-x} dx = k\int_0^1 (2x - x^2)\,dx = k\left(x^2 - \tfrac{1}{3}x^3\right)\Big|_0^1$

$= \tfrac{2}{3}k = 1 \Rightarrow k = \tfrac{3}{2}$

(c) $f_X(x) \underset{0 \le x \le 1}{=} \int_0^{2-x} \tfrac{3}{2}x\,dy = \tfrac{3}{2}xy\Big|_0^{2-x} = \tfrac{3}{2}x(2-x)$

$$= \begin{cases} 3x - \tfrac{3}{2}x^2, & 0 \le x \le 1 \\ 0, & \text{else} \end{cases}$$

Check: $\int_0^1 (3x - \tfrac{3}{2}x^2)\,dx = 1$ ✓

$f_Y(y) \underset{0 \le y \le 1}{=} \int_0^1 \tfrac{3}{2}x\,dx = \tfrac{3}{4}x^2\Big|_0^1 = \tfrac{3}{4}$

$f_Y(y) \underset{1 \le y \le 2}{=} \int_0^{2-y} \tfrac{3}{2}x\,dx = \tfrac{3}{4}x^2\Big|_0^{(2-y)} = \tfrac{3}{4}(2-y)^2 = 3 - 3y + \tfrac{3}{4}y^2$

$$\Rightarrow f_Y(y) = \begin{cases} \tfrac{3}{4}, & 0 \le y \le 1 \\ 3 - 3y + \tfrac{3}{4}y^2, & 1 \le y \le 2 \\ 0, & \text{else} \end{cases}$$

Check: $\int_{-\infty}^{\infty} f_Y(y)\,dy = 1$ ✓

(d)

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \dfrac{3/2\,x}{3/4} = 2x, & 0 \le x \le 1,\ 0 \le y \le 1 \\[2ex] \dfrac{3/2\,x}{3-3y+3/4\,y^2}, & 0 \le x \le 2-y,\ 1 \le y \le 2 \\[2ex] 0, & \text{else} \end{cases}$$

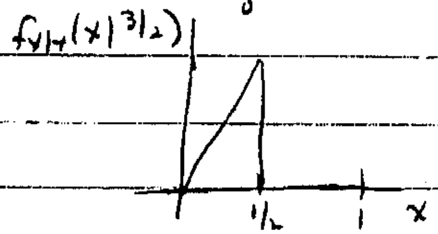
<u>Check:</u> $\int_0^1 2x\,dx = 1$, $\int_0^{2-y} \frac{3/2\,x}{3-3y+3/4\,y^2}\,dx = 1$ ✓

(Integrates to 1 for any y)

(e)

- $$f_{X|Y}(x|3/2) = \begin{cases} \dfrac{3/2\,x}{3-3(3/2)-3/4(3/2)^2} = 8x & 0 \le x \le 1/2 \\[2ex] 0, & \text{else} \end{cases}$$

(<u>Check:</u> $\int_0^{1/2} 8x\,dx = 1$ ✓)



- choose $\hat{X} = 0.4$ to cover $(0.3, 0.5)$

2) (a)

$$P(1 < X \le 4) = P(X \le 4) - P(X \le 1)$$
$$= \Phi\left(\frac{4-2}{2}\right) - \Phi\left(\frac{1-2}{2}\right)$$
$$= \Phi(1) - \Phi(-1/2) = \Phi(1) - (1 - \Phi(1/2))$$
$$= \Phi(1) + \Phi(1/2) - 1$$

(b)

$$f_{X|\{X \ge 4\}}(x) = \begin{cases} \dfrac{f_X(x)}{P(X \ge 4)} = \dfrac{1/\sqrt{2\pi \cdot 4}\, e^{-(x-2)^2/2\cdot 4}}{1 - \Phi(1)}, & x \ge 4 \\ 0, & \text{else} \end{cases}$$

(c) Many ways. Here is one:

$$\Phi(x) = 1/2 + \int_0^x \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$$
$$v = u/\sqrt{2} \quad \overset{\downarrow}{=} 1/2 + \int_0^{x/\sqrt{2}} \frac{1}{\sqrt{2\pi}} e^{-v^2} dv \cdot \sqrt{2}$$
$$= 1/2 + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{2}} e^{-v^2} dv$$
$$= 1/2 + 1/2 \operatorname{erf}(x/\sqrt{2})$$

$$\Rightarrow P(1 < X \le 4) = 1/2 + 1/2 \operatorname{erf}(1/\sqrt{2})$$
$$+ 1/2 + 1/2 \operatorname{erf}(1/2\sqrt{2})$$
$$- 1$$
$$= 1/2 \operatorname{erf}(1/\sqrt{2}) + 1/2 \operatorname{erf}(1/2\sqrt{2})$$

3) (a)

$$P(\text{roll}=1) = P(\text{roll}=1 \mid \text{Die }1)P(\text{Die }1) + P(\text{roll}=1 \mid \text{Die }2)P(\text{Die }2) + P(\text{roll}=1 \mid \text{Die }3)P(\text{Die }3)$$

$$= \tfrac{1}{2}\cdot\tfrac{1}{3} + \tfrac{1}{3}\cdot\tfrac{1}{3} + \tfrac{1}{4}\cdot\tfrac{1}{3}$$

$$= \tfrac{1}{6} + \tfrac{1}{9} + \tfrac{1}{12} = \tfrac{6}{36} + \tfrac{4}{36} + \tfrac{3}{36} = \tfrac{13}{36}$$

(b) $$P(3\ 1\text{'s}) = P(3\ 1\text{'s} \mid \text{Die }1)P(\text{Die }1) + P(3\ 1\text{'s} \mid \text{Die }2)P(\text{Die }2) + P(3\ 1\text{'s} \mid \text{Die }3)P(\text{Die }3)$$

$$= (\tfrac{1}{2})^3\cdot\tfrac{1}{3} + (\tfrac{1}{3})^3\cdot\tfrac{1}{3} + (\tfrac{1}{4})^3\cdot\tfrac{1}{3}$$

(c) $$P(X=x_j, Y=y_k) = P(Y=y_k \mid X=x_j)\,P(X=x_j)$$

y_k \ x_j	1	2	3
4	0	0	$\frac{1}{12}$
3	0	$\frac{1}{9}$	$\frac{1}{12}$
2	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{12}$
1	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{12}$

$X > Y$

(d) See circles above: $P(X>Y) = \frac{1}{12} + \frac{1}{12} + \frac{1}{9} = \frac{5}{18}$

(e) (Sum rows) $$p_Y(y_k) = \begin{cases} \frac{13}{36}, & y_k = 1, 2 \\ \frac{7}{36}, & y_k = 3 \\ \frac{3}{36}, & y_k = 4 \\ 0, & \text{else} \end{cases}$$

(f) $$p_{X|Y}(x_j|1) = \frac{P(X=x_j \cap Y=1)}{P(Y=1)} = \begin{cases} 6/13, & x_j=1 \\ 4/13, & x_j=2 \\ 3/13, & x_j=3 \\ 0, & \text{else} \end{cases}$$

(g) NO

$$p_{X|Y}(x_j|1) \neq p_X(x_j) = \begin{cases} 1/3, & x_j=1 \\ 1/3, & x_j=2 \\ 1/3, & x_j=3 \\ 0, & \text{else} \end{cases}$$