ECE 603 - Probability and Random Processes, Fall 2015

Midterm Exam #2

November 18, 7:00-9:00pm

Ag. Engineering, Room 119

Overview

- The exam consists of four problems for 130 points. The points for each part of each problem are given in brackets - you should spend your **two hours** accordingly.
- The exam is closed book, but you are allowed **two page-sides** of notes. Calculators are not allowed. I will provide all necessary blank paper.

Testmanship

- **Full credit will be given only to fully justified answers.**
- Giving the steps along the way to the answer will not only earn full credit but also maximize the partial credit should you stumble or get stuck. If you get stuck, attempt to neatly define your approach to the problem and why you are stuck.
- If part of a problem depends on a previous part that you are unable to solve, explain the method for doing the current part, and, if possible, give the answer in terms of the quantities of the previous part that you are unable to obtain.
- Start each problem on a new page. Not only will this facilitate grading but also make it easier for you to jump back and forth between problems.
- If you get to the end of the problem and realize that your answer must be wrong, be sure to write "this must be wrong because . . . " so that I will know you recognized such a fact.
- Academic dishonesty will be dealt with harshly - the *minimum penalty* will be an "F" for the course.

Facts you *might* find useful as you work the exam:

$$\int \ln x \, dx = x(\ln x - 1) + \text{constant}$$

$$\frac{d}{dx}\int_{g_1(x)}^{g_2(x)} h(x,y)dy = \frac{dg_2(x)}{dx}\cdot h(x,g_2(x)) - \frac{dg_1(x)}{dx}\cdot h(x,g_1(x)) + \int_{g_1(x)}^{g_2(x)} \frac{\partial h(x,y)}{\partial x}dy$$

1. The random variables X and Y have joint probability density function (pdf) given by:

$$f_{X,Y}(x,y) = \begin{cases} c, & 1 \le x \le 2, \quad -2x \le y \le 2x \\ 0, & \text{else} \end{cases}$$

[7] (a) Find c.

[8] (b) Find $f_Y(y)$, the marginal probability density function (pdf) of Y.

[5] (c) Find $P(|Y| < X^2)$.

[5] (d) Find $f_{X|Y}(x|y)$, the conditional probability density function of X given Y. For your limits (**which you should not forget**), put y between constant limits (**not dependent on** x), and then give the limits for x.

[5] (e) Are X and Y independent?

[5] (f) Suppose you measure $Y = 3.0$ and you want to estimate the value of X. Select a "reasonable" value of X for your estimate (and justify).

2. Suppose that the random variables X and Y are jointly Gaussian with $E[X] = 1$, $\text{Var}(X) = 16$, $E[Y] = -1$, $\text{Var}(Y) = 9$, and $\rho_{X,Y} = 1/3$.

[5] (a) Suppose $W = X + 5$. Find $f_W(w)$, the probability density function (pdf) of W.

[5] (b) Suppose $V = 3X + 2$. Find $f_V(v)$, the probability density function (pdf) of V.

[10] (c) Suppose $Z = X - 3Y$. Find $f_Z(z)$, the probability density function (pdf) of Z.

3. *Convergence Problems:*

[8] (a) Let the probability space $(\Omega, \mathcal{A}, P)$ be given by $\Omega = [0,1]$, $\mathcal{A} = \mathcal{B}$ (restricted to $[0,1]$, of course), and $P((a,b)) = b - a$. Let $X_n(\omega) = \omega^3/\sqrt{n}$. Determine whether the sequence $\{X_n\}$ converges, and, if so, to what and in what ways? Consider almost sure convergence, mean square

convergence, convergence in probability, and convergence in distribution. You can do them in any order that you want, and you can use one type of convergence to imply another when appropriate.

[8] (b) Let the probability space $(\Omega, \mathcal{A}, P)$ be given by $\Omega = [0,1]$, $\mathcal{A} = \mathcal{B}$ (restricted to $[0,1]$, of course), and $P((a,b)) = b - a$. Let

$$X_n(\omega) = \begin{cases} \omega/n, & \omega \text{ irrational} \\ 1, & \omega \text{ rational} \end{cases}$$

Determine in what ways the sequence $\{X_n\}$ converges to $X = 0$. Consider almost sure convergence, mean square convergence, convergence in probability, and convergence in distribution. You can do them in any order that you want, and you can use one type of convergence to imply another when appropriate.

[8] (c) Consider the probability space $(S, \mathcal{A}, P)$, with $S = [0,1]$ and $\mathcal{A} = \mathcal{B}$ (restricted to $[0,1]$,of course), and $P(\cdot)$ defined as follows:

$$P((a,b)) = \begin{cases} ((b-a) - \frac{1}{2}(b^2 - a^2)), & 0 \le a < b \le \frac{1}{4} \\ ((b-a) - \frac{1}{2}(b^2 - a^2)) & \frac{1}{4} \le a < b \le 1 \\ \frac{1}{2} + ((b-a) - \frac{1}{2}(b^2 - a^2)), & a < \frac{1}{4} < b \le 1 \end{cases}$$

Let

$$X_n(\omega) = \begin{cases} \omega/n, & \omega \text{ irrational} \\ 1, & \omega \text{ rational} \end{cases}$$

Determine in what ways the sequence $\{X_n\}$ converges to $X = 0$. Consider almost sure convergence, mean square convergence, convergence in probability, and convergence in distribution. You can do them in any order that you want, and you can use one type of convergence to imply another when appropriate.

[8] (d) Let the probability space $(\Omega, \mathcal{A}, P)$ be given by $\Omega = [0,1]$, $\mathcal{A} = \mathcal{B}$ (restricted to $[0,1]$, of course), and $P((a,b)) = b - a$. Let

$$X_n(\omega) = \frac{\lfloor \omega \cdot n \rfloor}{n}$$

where $\lfloor x \rfloor$ is the largest integer less than x. Another way of stating $X_n(\omega)$: it is "rounding down" ω to the nearest c/n less than ω, where c is an integer. Determine whether the sequence $\{X_n\}$ converges, and, if so, to what and in what ways? Consider almost sure convergence, mean square convergence, convergence in probability, and convergence in distribution. You can do them in any order that you want, and you can use one type of convergence to imply another when appropriate.

[8] (e) Let the probability space $(\Omega, \mathcal{A}, P)$ be given by $\Omega = [0,1]$, $\mathcal{A} = \mathcal{B}$ (restricted to $[0,1]$, of course), and $P((a,b)) = b - a$. Let $X_n(\omega) = \omega$, for $n = 0, 2, 4, 6, 8, \ldots$ (i.e. n even). Let $X_n(\omega) = 1 - \omega$, for $n = 1, 3, 5, 7, \ldots$ (i.e. n odd). Determine whether the sequence $\{X_n\}$ converges, and, if so, to what and in what ways? Consider almost sure convergence, mean square convergence, convergence in probability, and convergence in distribution. You can do them in any order that you want, and you can use one type of convergence to imply another when appropriate.

4. Let X be a random variable with probability density function given by:

$$f_X(x) = \begin{cases} c, & 0 \le x \le 5 \\ 0, & \text{else} \end{cases}$$

where c is a constant.

[12] (a) Perform the following (one sentence of justification is fine in each case):

- Find the constant c.
- Find the probability density function (pdf) of $X + 2$.
- Find the probability density function (pdf) of $-X$.
- Find the probability density function (pdf) of $X + X$.

Now, suppose that, given $X = x$, Y is uniformly distributed between 0 and x.

[8] (b) Find $f_Y(y)$, the probability density function of Y.

[15] (d) Let $Z = X + Y$. Find $f_Z(z)$, the probability density function of Z.