ECE 314 - Intro to Probability and Random Processes (Spring 2013)

Midterm Exam #2

Wednesday, April 3rd, 7:00-9:00pm, ELAB II Auditorium

Overview

- The exam consists of four problems for 100 points. The points for each part of each problem are given in brackets - you should spend your **two hours** accordingly.
- The exam is closed book, but you are allowed **two page-sides** of notes. Calculators are **not** allowed. I will provide all necessary blank paper.

Testmanship

- **Full credit will be given only to fully justified answers.**
- Giving the steps along the way to the answer will not only earn full credit but also maximize the partial credit should you stumble or get stuck. If you get stuck, attempt to neatly define your approach to the problem and why you are stuck.
- If part of a problem depends on a previous part that you are unable to solve, explain the method for doing the current part, and, if possible, give the answer in terms of the quantities of the previous part that you are unable to obtain.
- Start each problem on a new page. Not only will this facilitate grading but also make it easier for you to jump back and forth between problems.
- If you get to the end of the problem and realize that your answer must be wrong (e.g. a negative probability, a pmf that does not sum to one), be sure to write "this must be wrong because . . . " so that I will know you recognized such a fact.
- Academic dishonesty **will** be dealt with **harshly** - the *minimum penalty* will be an "F" for the course.

1. The random variable X has probability density function (pdf) $f_X(x)$ given by:

$$f_X(x) = \begin{cases} c_1(2-x), & 0 \le x \le 2 \\ 0, & \text{else} \end{cases}$$

[5] (a) Sketch $f_X(x)$ **and** find c_1.

[5] (b) Find $P(X \ge 1)$.

[7] (c) Suppose $Y = X^2$. Find the probability density function $f_Y(y)$ of Y.

[8] (d) Suppose that I want to generate the random variable X using what I have in my backpack: a (fair) coin, a deck of cards, a six-sided die, a ten-sided die, and a cheap calculator. Complete the following steps.

- Describe how to use one (or more, if needed) of the objects in my backpack to generate a random variable U that is uniformly distributed on the *interval* $[0, 1]$.
- Find a function $g(\cdot)$ such that, when I set $X = g(U)$, the resulting random variable X has probability density function $f_X(x)$.

2. The random variable Z has probability density function (pdf) $f_Z(z)$ given by:

$$f_Z(z) = 0.5\ \delta\left(z - \frac{3}{2}\right) + h(z)$$

where

$$h(z) = \begin{cases} c_2(2-z), & 0 \le z \le 2 \\ 0, & \text{else} \end{cases}$$

(*Hint: The above might look weird, but just draw $h(z)$ and then add the delta function in $f_Z(z)$ to your sketch.*)

[5] (a) Sketch $f_Z(z)$ **and** find c_2.

[7] (b) Find $E[Z^2]$.

[8] (c) Consider the following function (a quantizer) $Q(\cdot)$:

$$Q(t) = \begin{cases} 1/5, & 0 \le t < 2/5 \\ 3/5, & 2/5 \le t < 4/5 \\ 1, & 4/5 \le t < 6/5 \\ 7/5, & 6/5 \le t < 8/5 \\ 9/5, & 8/5 \le t \le 2 \\ 0, & \text{else} \end{cases}$$

Let $V = Q(Z)$. Find $F_V(v) = P(V \le v)$, the cumulative distribution function (CDF) of V. **NOTES:**

- **You do not need to fully evaluate the integrals, which is tedious; rather, since the portions of the integrals involving the $\delta(\cdot)$ function are easy, just do those parts (In other words, make sure there are no $\delta(\cdot)$'s in your answer, but you can leave integrals of $h(z)$.)**
- **If you cannot get part (a) of this question, find $F_V(v)$ when $V = Q(X)$ for the random variable X from Problem 1, for substantial partial credit.**

3. I am trying to make an accurate measurement and, of course, want to reduce the error when I do such. I have two measurement devices available for use: $\mathcal{X}$ and $\mathcal{Y}$. The error when I use device $\mathcal{X}$ has probability density function given by:

$$f_X(x) = \frac{1}{\sqrt{8\pi}} e^{-\frac{(x-2)^2}{8}}$$

and the error when I use device $\mathcal{Y}$ has probability density function given by:

$$f_Y(y) = \frac{1}{\sqrt{32\pi}} e^{-\frac{(y+2)^2}{32}}$$

[5] (a) Suppose I use device $\mathcal{X}$. Find the probability that the error is between -1 and 1; that is, find $P(-1 < X < 1)$. (Write your answer in terms of the $\Phi(\cdot)$ function defined in class.)

[7] (b) Suppose I grab a device at random (i.e. equally likely). Find the probability that the resulting error Z satisfies $P(-1 < Z < 1)$. (Write your answer in terms of the $\Phi(\cdot)$ function defined in class.)

[8] (c) You start work at your first company after graduation and find that they do not have the $\Phi(\cdot)$ function implemented; rather, they have only the "Acme Corporation Function"

$$\mathcal{A}(x) = \int_x^\infty \frac{\sqrt{\pi}}{2} e^{\frac{-u^2}{4}} du$$

Write $\Phi(x)$ in terms of $\mathcal{A}(\cdot)$.

4. (*Note: For simplicity, this problem uses a different probability mass function than what you might expect [Poisson] for the application and has* **nothing** *to do with Poisson random variables.*)

The number of cars X passing the classroom window in a one minute interval has probability mass function (pmf)

$$p_X(x_k) = P(X = x_k) = \begin{cases} c/9, & x_k = 2 \\ 2c/9, & x_k = 3 \\ 3c/9, & x_k = 4 \\ 2c/9, & x_k = 5 \\ 0 & \text{else} \end{cases}$$

Student $\mathcal{Y}$ is supposed to be counting the cars, but he/she is distracted and thus makes errors. In fact, **given** $X = x_k$, the number of cars recorded is equally likely (i.e. probability of 1/3 each) to be $x_k - 1$, x_k, or $x_k + 1$. In other words, given the number of cars, the student is equally likely to record the correct number of cars, one more, or one less. Let Y be the number of cars recorded by student $\mathcal{Y}$.

[5] (a) Find c.

[10] (b) Find the joint probability mass function $p_{X,Y}(x_k, y_k) = P(X = x_k, Y = y_k)$. (You can give your answer in any form you want: a table, sticks with balls on top, etc.)

[5] (c) Find the marginal probability mass function for Y, $p_Y(y_k) = P(Y = y_k)$.

[10] (d) Suppose that student $\mathcal{Y}$ records that he/she saw 5 cars and we are interested in trying to guess the actual number of cars given that observation:

- Find **and plot** the conditional probability mass function $p_{X|Y}(x_k|5)$ of X given $Y = 5$.
- What is the most probable value for X given $Y = 5$?

[5] (e) Are X and Y independent?