ECE 314 - Intro to Probability and Random Processes (Spring 2012)

Midterm Exam #2

Wednesday, April 11th, 7:00-9:00pm, Herter Room 227

Overview

- The exam consists of three problems for 100 points. The points for each part of each problem are given in brackets - you should spend your **two hours** accordingly.
- The exam is closed book, but you are allowed **two page-sides** of notes. Calculators are **not** allowed. I will provide all necessary blank paper.

Testmanship

- **Full credit will be given only to fully justified answers.**
- Giving the steps along the way to the answer will not only earn full credit but also maximize the partial credit should you stumble or get stuck. If you get stuck, attempt to neatly define your approach to the problem and why you are stuck.
- If part of a problem depends on a previous part that you are unable to solve, explain the method for doing the current part, and, if possible, give the answer in terms of the quantities of the previous part that you are unable to obtain.
- Start each problem on a new page. Not only will this facilitate grading but also make it easier for you to jump back and forth between problems.
- If you get to the end of the problem and realize that your answer must be wrong (e.g. a negative probability, a pmf that does not sum to one), be sure to write "this must be wrong because . . . " so that I will know you recognized such a fact.
- Academic dishonesty **will** be dealt with **harshly** - the *minimum penalty* will be an "F" for the course.

1. The random variables X and Y have joint pdf given by:

$$f_{X,Y}(x,y) = \begin{cases} kx, & 0 \le x \le 1, \quad 0 \le y \le 2-x \\ 0, & \text{otherwise} \end{cases}$$

[5] (a) Sketch the region in the (x,y) plane where $f_{X,Y}(x,y) > 0$.

[5] (b) Find k.

[10] (c) Find $f_X(x)$ and $f_Y(y)$, the marginal probability density functions (pdfs) of X and Y, respectively.

[7] (d) Find $f_{X|Y}(x|y)$, the conditional probability density function of X given Y. For your limits (**which you should not forget**), put y between constant limits, and then give the limits for x.

[8] (e) Suppose you measure $Y = 1.5$ and you want to estimate the value of X within some small tolerance:

- Find and sketch $f_{X|Y}(x|1.5)$.
- Pick $\hat{X}$ such that $|\hat{X} - X| < 0.1$ with the highest probability.

2. Let X be Gaussian with mean $\mu = 2$ and variance $\sigma^2 = 4$. For parts (a) and (b), you can leave your answers in terms of the $\Phi(\cdot)$ function, which recall is defined as:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du$$

but is only generally tabulated for $x \ge 0$. In other words, make sure your argument of $\Phi(\cdot)$ is positive.

[7] (a) Find $P(1 < X \le 4)$.

[5] (b) Find $f_{X|\{X\ge 4\}}(x)$; that is, find the density function of X given $X \ge 4$.

[8] (c) Suppose you do not have the integration table for $\Phi(x)$, but rather a table for $\text{erf}(x)$, where:

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-u^2} du$$

Repeat part (a), but give your answer in terms of $\text{erf}(\cdot)$ instead of $\Phi(x)$.

3. I have three dice in my pocket that behave as follows when rolled individually:

   - **Die 1:** Equally likely to result in a 1 or 2.
   - **Die 2:** Equally likely to result in a 1,2, or 3.
   - **Die 3:** Equally likely to result in a 1,2,3, or 4.

   [7] (a) I choose one die at random (i.e. each die has a one-third chance of being selected) and roll it once. Find the probability that the result of the roll is a "1".

   [8] (b) I choose one die at random (i.e. each die has a one-third chance of being selected) and roll that die three times. Find the probability that I get "1" all three times. (**No need to work out messy fractions.**)

   Now, I return to selecting a die at random (i.e. each die has a one-third chance of being selected) and rolling the die **one time**. Let:

   - X: The number of the die chosen (i.e. 1, 2, or 3).
   - Y: The result of the roll.

   [10] (c) Find the joint probability mass function (pmf) $p_{X,Y}(x_j, y_k) = P(X = x_j, Y = Y_k)$. You can display the result any way you want: a table with the entries, a plot of "sticks with balls", a plot with circles with probability labels, etc.

   [5] (d) Find $P(X > Y)$, the probability that X is greater than Y.

   [5] (e) Find the marginal probability distribution $p_Y(y_k) = P(Y = y_k)$.

   [5] (f) Suppose you are interested in X but only observe Y. You measure $Y = 1$. Find $p_{X|Y}(x_j|1)$.

   [5] (g) Are X and Y independent?