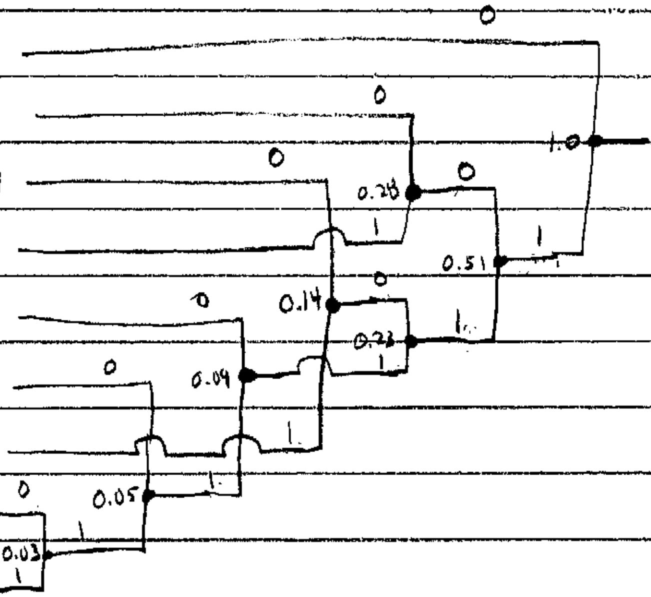
Midterm 1 Solutions

ECE 564/645 Spring 2014

1) (a)

x	$P(x)$	code	$P(x)\,l(x)$
AA	0.49	0	0.49
AB	0.14	100	0.42
AC	0.07	1100	0.28
BA	0.14	101	0.42
BB	0.04	1110	0.16
BC	0.02	11110	0.10
CA	0.07	1101	0.28
CB	0.02	111110	0.12
CC	0.01	111111	0.06
			2.33



$$R = 2.33/2 = 1.165 \text{ bits/symbol}$$

$$H(X) \le R_{Huff} \le H(X) + 1/N$$

$$1.16 \le 1.165 \le 1.66$$

(b) (i) <u>not</u> prefix-free: 111 is a prefix of 1110

(ii) <u>not</u> optimal: I could get a lower rate lossless code by interchanging 1100 and 101

(iii) <u>not</u> optimal: a better code is

$$\{00, 01, 100, 101, 1100, 1101, 1110, 1111\}$$

2) (a)

$$\tilde{\phi}_1(t) = s_1(t)$$

$$\int_{-\infty}^{\infty} s_1^2(t)\,dt = \int_0^2 1^2\,dt = 2$$

$$\Rightarrow \phi_1(t) = s_1(t)/\sqrt{2}$$

$$v_2(t) = s_2(t) - \left(\int_0^{T_s} s_2(t)\phi_1(t)\,dt\right)\phi_1(t)$$

$0 \le t \le 2$
$$= (2-t) - \underbrace{\left(\int_0^2 (2-t)\cdot 1/\sqrt{2}\,dt\right)}_{\frac{(2t - 1/2 t^2)}{\sqrt{2}}\Big|_{t=0}^{t=2} = \sqrt{2}}\phi_1(t)$$

$$= (2-t) - \sqrt{2}\, s_1(t)/\sqrt{2} = 1 - t$$

$$\int_0^2 (1-t)^2\,dt = \int_0^2 (1 - 2t + t^2)\,dt$$

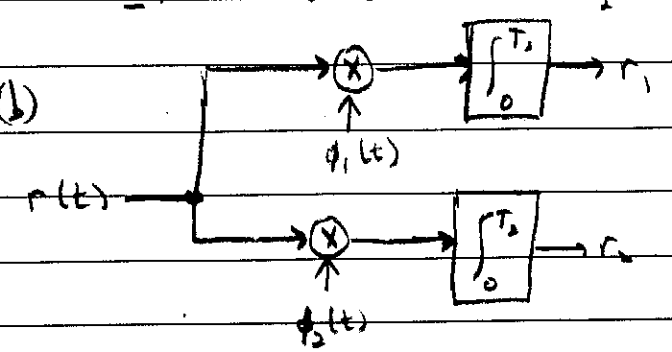
$$= \left(t - t^2 + 1/3\, t^3\right)\Big|_{t=0}^{2}$$

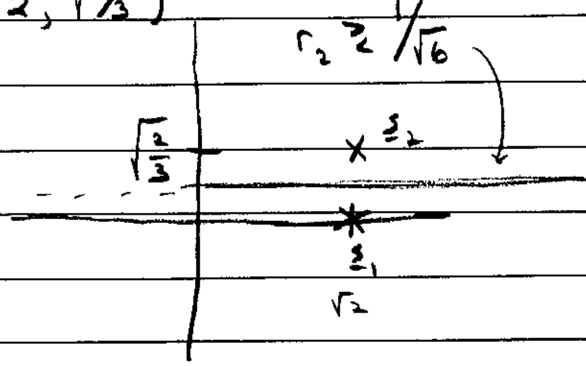
$$= 2/3$$

$$\Rightarrow \phi_2(t) = \begin{cases} \sqrt{\frac{3}{2}}(1-t), & 0 \le t \le 2 \\ 0, & \text{else} \end{cases}$$

$$\underline{s}_1 = (\sqrt{2}, 0)^T \qquad \underline{s}_2 = \left(\sqrt{2}, \sqrt{2/3}\right)$$

(b)





(c)

$$P(\varepsilon) = Q\left(d/\sqrt{2N_0}\right)$$

$$= Q\left(\sqrt{2}/3 \Big/ \sqrt{2N_0}\right)$$

$$= Q\left(1/\sqrt{3N_0}\right)$$

$$E_s = \tfrac{1}{2}(\sqrt{2})^2 + \tfrac{1}{2}\left((\sqrt{2})^2 + (\sqrt{2}/3)^2\right)$$

$$= 1 + \tfrac{1}{2}(2 + 2/3)$$

$$= 1 + 4/3 = 7/3$$

$$P(\varepsilon) = Q\left(\sqrt{\frac{3/7 \cdot 7/3}{3N_0}}\right)$$

$$= Q\left(\sqrt{\frac{1}{7}\frac{E_s}{N_0}}\right)$$

$10 \log_{10} 14$ <u>worse</u> than BPSK

3)

(a)

$$\phi_1(t) = \sqrt{\tfrac{2}{T_s}}\cos(2\pi f_c t)$$

$$\phi_2(t) = \sqrt{\tfrac{2}{T_s}}\sin(2\pi f_c t) \quad \leftarrow \text{There are other choices}$$

$$\left(\text{like } \phi_2(t) = -\sqrt{\tfrac{2}{T_s}}\sin(2\pi f_c t)\right)$$

$$\underline{s}_1 = (0,0)^T$$

$$\underline{s}_2 = \left(2\sqrt{T_s/2},\, 0\right)^T$$

$$s_3(t) = 2\sqrt{2}\left(\cos(2\pi f_c t)\,\overset{1/\sqrt{2}}{\cancel{\cos(\pi/4)}} - \sin(2\pi f_c t)\,\overset{1/\sqrt{2}}{\cancel{\sin(\pi/4)}}\right)$$

$$= 2\cos(2\pi f_c t) - 2\sin(2\pi f_c t)$$

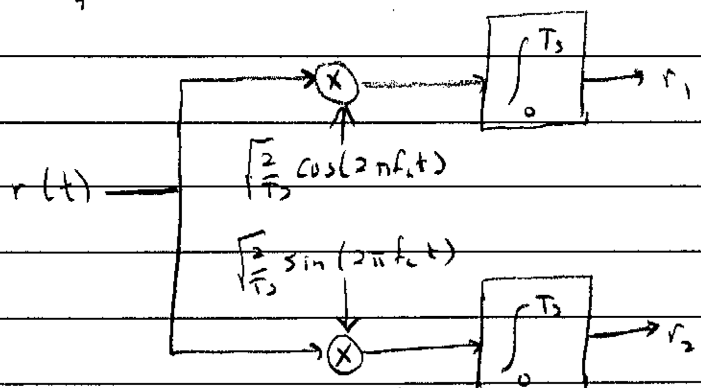
$$\underline{s}_3 = \left(2\sqrt{T_s/2},\, -2\sqrt{T_s/2}\right)^T$$

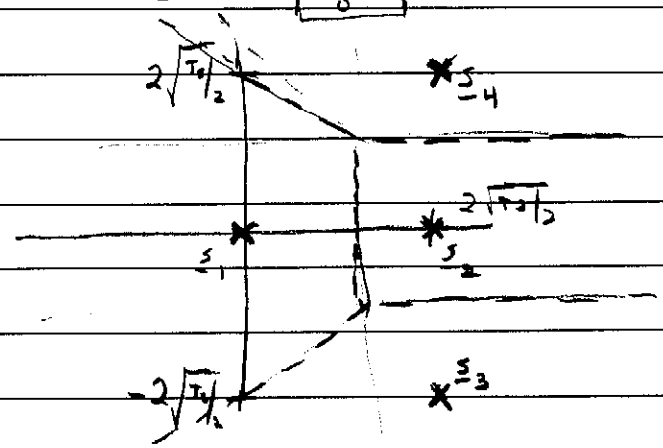
$$\underline{s}_4(t) = 2\sqrt{2}\left(\cos(2\pi f_c t)\,\overset{1/\sqrt{2}}{\cancel{\cos(\pi/4)}} - \sin(2\pi f_c t)\,\overset{-1/\sqrt{2}}{\cancel{\sin(-\pi/4)}}\right)$$

$$= 2\cos(2\pi f_c t) + 2\sin(2\pi f_c t)$$

$$\underline{s}_4 = \left(2\sqrt{T_s/2},\, 2\sqrt{T_s/2}\right)^T$$

(b)





(c)

Go to easier units!

$s_1 = (0,0)^T$

$s_2 = (1,0)^T$

$s_3 = (1,1)^T$

$s_4 = (1,-1)^T$

d_{ij} $j \backslash i$	1	2	3	4
1	–	1	$\sqrt{2}$	$\sqrt{2}$
2	1	–	1	1
3	$\sqrt{2}$	1	–	2
4	$\sqrt{2}$	1	2	–

$$P(E) \le \frac{1}{4} \sum_{i=1}^{4} \sum_{j \ne i} Q\left(d_{ij}/\sqrt{2N_0}\right)$$

$$= \tfrac{1}{4}\left(6\,Q\left(1/\sqrt{2N_0}\right) + 4\,Q\left(\sqrt{2}/\sqrt{2N_0}\right) + 2\,Q\left(2/\sqrt{2N_0}\right)\right)$$

$$= \frac{3}{2} Q\left(1/\sqrt{2N_0}\right) + Q\left(\sqrt{1/N_0}\right) + \frac{1}{2} Q\left(\sqrt{2/N_0}\right)$$

$$E_s = \frac{1}{4}\left(1^2 + (1^2+1^2) + (1^2 + (-1)^2)\right)$$

$$= 5/4$$

$$P(E) \le \frac{3}{2} Q\left(\sqrt{\frac{4/5 \cdot 5/4}{2N_0}}\right) + Q\left(\sqrt{\frac{4/5 \cdot 5/4}{N_0}}\right) + \frac{1}{2} Q\left(\sqrt{\frac{2 \cdot 4/5 \cdot 5/4}{N_0}}\right)$$

$$= \frac{3}{2} Q\left(\sqrt{2/5\, E_s/N_0}\right) + Q\left(\sqrt{4/5\, E_s/N_0}\right) + \frac{1}{2} Q\left(\sqrt{8/5\, E_s/N_0}\right)$$

$E_s = 2E_b$

$$= \frac{3}{2} Q\left(\sqrt{4/5\, E_b/N_0}\right) + Q\left(\sqrt{8/5\, E_b/N_0}\right) + \frac{1}{2} Q\left(\sqrt{16/5\, E_b/N_0}\right)$$

(d) High SNR. Good, because we can simulate at low SNR.

(e) QPSK: $Q\left(\sqrt{2E_b/N_0}\right)$ $\quad 2/(4/5) = 5/2$ $\quad 10 \log_{10} 5/2$ dB worse!

4) (a) Equally likely, thus, want: choose s_1 if

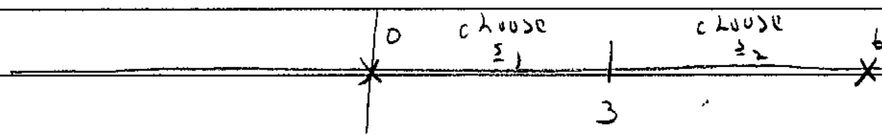
$$p(r|s_1) \geq p(r|s_2)$$

$$\frac{1}{2} e^{-|r-s_1|} \geq \frac{1}{2} e^{-|r-s_2|}$$

$$\cancel{\frac{1}{2}} e^{-|r|} \geq \cancel{\frac{1}{2}} e^{-|r-6|}$$

$$-|r| \geq -|r-6|$$

$$|r| \leq |r-6|$$



(b)

$$P(E) = \frac{1}{2} P(r > 3 \mid s_1) + \frac{1}{2} P(r < 3 \mid s_2)$$

$$= \frac{1}{2} P(n > 3) + \frac{1}{2} P(n < -3)$$

$$= \frac{1}{2} \int_3^{\infty} \frac{1}{2} e^{-x} dx + \frac{1}{2} \int_{-\infty}^{-3} \frac{1}{2} e^{x} dx$$

$$= 1/2 \left(-1/2 e^{-x}\right)\Big|_3^{\infty} + 1/2 \left(1/2 e^{x}\right)\Big|_{-\infty}^{-3}$$

$$= 1/4 e^{-3} + 1/4 e^{-3}$$

$$= 1/2 e^{-3}$$

5)

(a) Under $s_1(t)$:

$$\int_0^1 4 \cdot 1 \, dt = 4$$

$$\int_1^2 4 \cdot 1 \, dt = 4$$

$s_1 : (4,4)^T$

Under $s_2(t)$:

$$\int_0^1 4(2-t)\,dt = 4(2t - \tfrac{1}{2}t^2)\Big|_0^1 = 6$$

$$\int_1^2 4(2-t)\,dt = 4(2t - \tfrac{1}{2}t^2)\Big|_1^2 = 2 - \tfrac{3}{2} = 2$$

Now, need to consider carefully the noise:

$$E[n_1] = \int_0^1 4\,E[n(t)]\,dt = 0$$

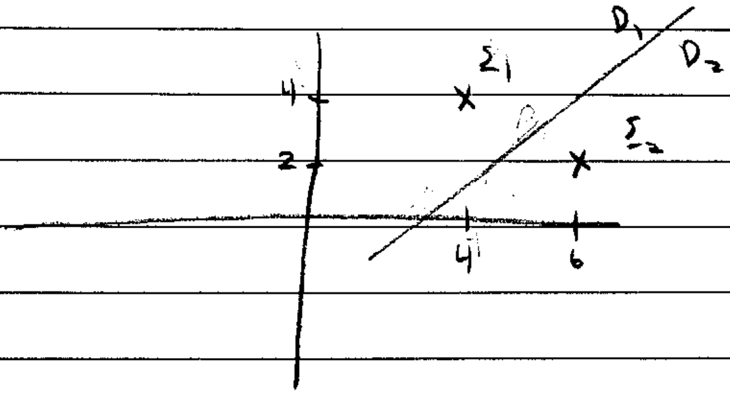
$$E[n_1^2] = \int_0^1\int_0^1 16\,\cancel{E[n(s)n(t)]}^{N_0/2\,\delta(s-t)}\,ds\,dt = 8N_0$$

$n_1 \sim N(0, 8N_0)$ Similarly, $n_2 \sim N(0, 8N_0)$

and $$E[n_1 n_2] = \int_0^1\int_1^2 16\,\cancel{E[n(t)n(s)]}^{N_0/2\,\delta(t,s)}\,dt\,ds$$

$$= 0$$

Thus, n_1 and n_2 are independent $\Rightarrow$ still an AWGN channel)



(b)

$$P(\varepsilon) = Q\left(d \big/ \sqrt{2(16N_0)}\right) = Q\left(d \big/ \sqrt{32N_0}\right)$$

$$d^2 = (2)^2 + (2)^2 = 8$$

$$P(\varepsilon) = Q\left(\sqrt{\frac{8}{32N_0}}\right) = Q\left(\sqrt{\frac{1}{4N_0}}\right) = Q\left(\sqrt{\frac{3/7 \cdot 7/3}{4N_0}}\right) = Q\left(\sqrt{\frac{3}{28}\frac{E_s}{N_0}}\right)$$

$E_s = 7/3$

from prob 2

(c) $10\log_{10} 56/3$ dB worse!