Midterm #1 Solutions

ECE 603 Fall 2009

1) (a) $P(S) = 3/4 + c\,(2^2 - 0^2) = 1 \Rightarrow c = 1/16$

$$P(1/2 < X < 1) = P((1/2, 1)) = 3/4 + \frac{1^2 - (1/2)^2}{16} = 3/4 + 3/64 = 51/64$$

(b)

$$P(2/3 \le X < 2) = P\left(\{X = 2/3\} \cup \{2/3 < X < 2\}\right)$$

$$\overset{3^{rd}\text{ axiom}}{=} P(X = 2/3) + P(2/3 < X < 2)$$

$$= P\left(\bigcap_{n=1}^{\infty} (2/3 - 1/n,\ 2/3 + 1/n)\right) + \frac{2^2 - (2/3)^2}{16}$$

$$= \lim_{n\to\infty} P(2/3 - 1/n,\ 2/3 + 1/n) + \frac{4 - 4/9}{16}$$

$$= \lim_{n\to\infty}\left(3/4 + \frac{(2/3 + 1/n)^2 - (2/3 - 1/n)^2}{16}\right) + \frac{32/9}{16}$$

$$= 3/4 + 2/9 = 35/36$$

-or-

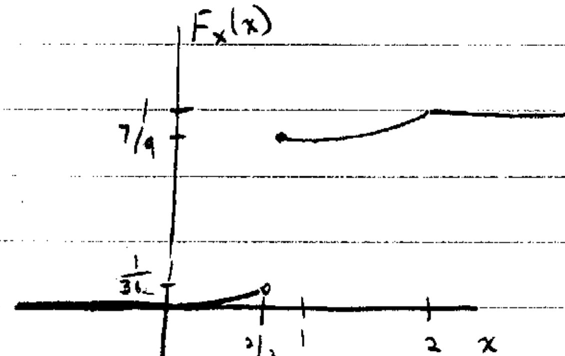
$$P(S) = P\left(\{0 < X < 2/3\} \cup \{2/3 \le X < 2\}\right)$$

$$= P((0, 2/3)) + P([2/3, 2))$$

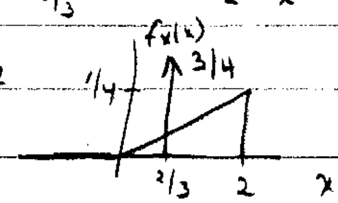
$$\Rightarrow P(2/3 \le X < 2) = 1 - P((0, 2/3)) = 1 - (2/3)^2/16 = 35/36$$

(c) $F_X(x) = P((-\infty, x])$

$$= \begin{cases} 0, & x \le 0 \\ x^2/16, & 0 < x < 2/3 \\ 3/4 + x^2/16, & 2/3 \le x \le 2 \\ 1, & x > 2 \end{cases}$$



$$f_X(x) = 3/4\,\delta(x - 2/3) + \begin{cases} x/8, & 0 \le x \le 2 \\ 0, & \text{else} \end{cases}$$



(d)

Since Y is a random variable: $S = \mathbb{R}$, $\mathcal{A} = \mathbb{B}$.

Now, need to find $P_Y((a,b))$. First, get $F_Y(y)$.

$$F_Y(y) = P(Y \le y)$$

$$= P(X^2 - 2 \le y)$$

$$= P(X^2 \le y+2)$$

$$= P(-\sqrt{y+2} \le X \le \sqrt{y+2})$$

$$= P(0 \le X \le \sqrt{y+2})$$

$$= F_X(\sqrt{y+2}) \qquad -2 \le y \le 2$$

$$= P_X((0, \sqrt{y+2}))$$

$$P_Y((a,b)) = \begin{cases} 1/16\,(b-a) & -2 \le a < b \le -14/9 \\ 1/16\,(b-a) & -14/9 \le a < b \le 2 \\ 3/4 + 1/16\,(b-a), & a < -14/9 < b \le 2 \end{cases}$$

2)

(a) Law of Total Prob

$$P(D_3) \overset{\text{Law of Total Prob}}{=} P(D_3 \mid D_2)P(D_2) + P(D_3 \mid \bar{D}_2)P(\bar{D}_2)$$

$$= 0.9 \cdot 0.8 + 0.5 \cdot 0.2$$

$$= 0.82$$

(b) $P(D_2 \mid D_3) = \dfrac{P(D_3 \mid D_2)P(D_2)}{P(D_3)} = \dfrac{0.9 \cdot 0.8}{0.82} = \dfrac{0.72}{0.82}$

(c) $P(D_1 \cap D_2 \cap D_3) \overset{\text{indep}}{=} P(D_2 \cap D_3)P(D_1)$

$$= P(D_3 \mid D_2)P(D_2)P(D_1)$$

$$= 0.9 \cdot 0.8 \cdot 0.8$$

$$= 0.576$$

(d) $P((D_1 \cap D_2 \cap \bar{D}_3) \cup (D_1 \cap \bar{D}_2 \cap D_3) \cup (\bar{D}_1 \cap D_2 \cap D_3))$

$$= P(\bar{D}_3 \mid D_2)P(D_2)P(D_1) + P(D_3 \mid \bar{D}_2)P(\bar{D}_2)P(D_1)$$

$$+ P(D_3 \mid D_2)P(D_2)P(\bar{D}_1)$$

$$= 0.1 \cdot 0.8 \cdot 0.8 + 0.5 \cdot 0.2 \cdot 0.8 + 0.9 \cdot 0.8 \cdot 0.2$$

$$= 0.064 + 0.08 + 0.144$$

$$= 0.288$$

3)

(b) $$P(\{L_1 \geq x+\ell\} \mid \{L_1 \geq x\}) = \frac{P(\{L_1 \geq x+\ell\} \cap \{L_1 \geq x\})}{P(\{L_1 \geq x\})}$$

$$= \frac{P(\{L_1 \geq x+\ell\})}{P(\{L_1 \geq x\})}$$

<u>memoryless</u> $$= e^{-(x+\ell)}/e^{-x} = e^{-\ell} = P(\{L_1 \geq \ell\})$$

(a) $$P(\{L_1 \geq 4\} \mid \{L_1 \geq 2\}) = \frac{P(\{L_1 \geq 4\} \cap \{L_1 \geq 2\})}{P(\{L_1 \geq 2\})}$$

$$= \frac{P(\{L_1 \geq 4\})}{P(\{L_1 \geq 2\})}$$

$$= \frac{\int_4^\infty e^{-x}\,dx}{\int_2^\infty e^{-x}\,dx} = e^{-4} = e^{-2}$$

(c) Let L: lifetime of lightbulb

$$P(L \geq 2) \overset{\text{Law of Total Prob}}{=} P(L \geq 2 \mid L = L_1)\,P(L = L_1) + P(L \geq 2 \mid L = L_2)\,P(L = L_2)$$
$$+ P(L \geq 2 \mid L = L_3)\,P(L = L_3)$$

$$= \left(\int_2^\infty e^{-x}\,dx\right)\cdot \tfrac{1}{3} + \left(\int_2^\infty 2e^{-2x}\,dx\right)\cdot \tfrac{1}{3} + \left(\int_2^\infty 3e^{-3x}\,dx\right)\cdot \tfrac{1}{3}$$

$$= \tfrac{1}{3}e^{-2} + \tfrac{1}{3}e^{-4} + \tfrac{1}{3}e^{-6}$$

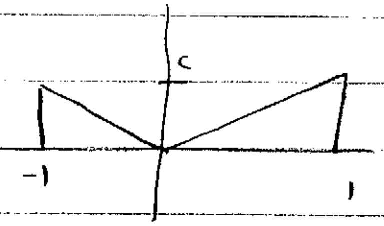
(d) $$P(L \geq 4 \mid L \geq 2) = \frac{P(\{L \geq 2\} \cap \{L \geq 4\})}{P(L \geq 2)}$$

$$= \frac{P(\{L \geq 4\})}{P(\{L \geq 2\})}$$

akin to (c) $$\overset{\downarrow}{=} \frac{\tfrac{1}{3}e^{-4} + \tfrac{1}{3}e^{-8} + \tfrac{1}{3}e^{-12}}{\tfrac{1}{3}e^{-2} + \tfrac{1}{3}e^{-4} + \tfrac{1}{3}e^{-6}}$$

(e) <u>not</u> <u>memoryless</u>. Combined distribution is not exponential. $\{L \geq 2\}$ carries information about which light!

4)



(a) $2\int_0^1 cx\,dx = 1 \Rightarrow 2\cdot c\cdot 1/2 = 1 \Rightarrow c = 1$

(b) $E[X] = \int_{-1}^{1} x|x|\,dx = \int_{-1}^{0} -x^2\,dx + \int_0^1 x^2\,dx$

$$= -1/3 + 1/3 = 0$$

(c) $P(X^2 \geq 1/2) = P(\{X \leq -1/\sqrt{2}\} \cup \{X \geq 1/\sqrt{2}\})$

$$= P(\{X \leq -1/\sqrt{2}\}) + P(\{X \geq 1/\sqrt{2}\})$$

$$= 2\int_{1/\sqrt{2}}^{1} x\,dx = x^2\Big|_{1/\sqrt{2}}^{1} = 1 - 1/2 = 1/2$$

(d) $E[3X^2 + 4X + 2] = 3E[X^2] + \cancelto{0}{4E[X]} + 2$

$$= 3\int_{-1}^{1} x^2|x|\,dx + 2$$

$$= 3\left(\int_{-1}^{0}(-x^3)\,dx + \int_0^1 x^3\,dx\right) + 2$$

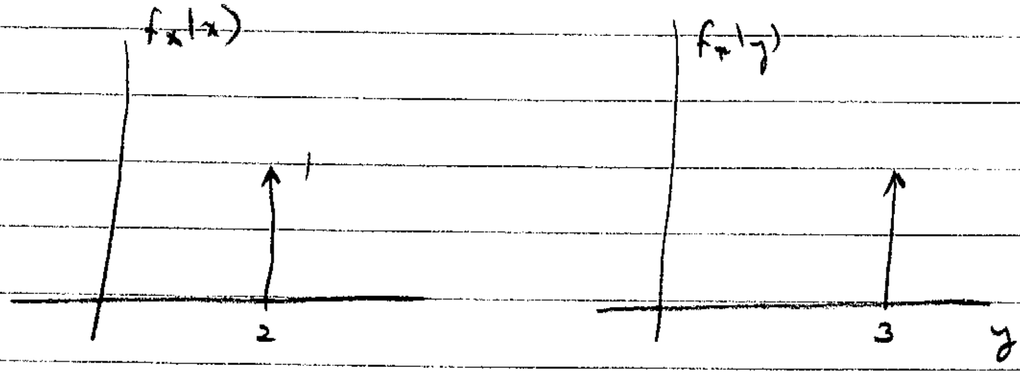
$$= 3\left(-1/4\,x^4\Big|_{-1}^{0} + 1/4\,x^4\Big|_0^1\right) + 2$$

$$= 3\left(1/4 + 1/4\right) + 2$$

$$= 7/2$$

5)

(a) Perhaps looking at the pdfs is easiest



X is 2 with probability 1

Y is 3 with probability 1

choose Y!

(b) $Z = X^2 + Y^2 = 13$ with probability 1

$$F_Z(z) = \begin{cases} 0, & z < 13 \\ 1, & z \geq 13 \end{cases}$$