ECE 564/645 - Digital Communication Systems (Spring 2014)

Midterm Exam #1

Monday, March 10th, 7:00-9:00pm, ELAB 303

Overview

- The exam consists of four (or five) problems for 100 (or 120) points. The points for each part of each problem are given in brackets - you should spend your **two hours** accordingly.
- The exam is closed book, but you are allowed **one page-side** of notes. Calculators are **not** allowed. I will provide all necessary blank paper.

Testmanship

- **Full credit will be given only to fully justified answers.**
- Giving the steps along the way to the answer will not only earn full credit but also maximize the partial credit should you stumble or get stuck. If you get stuck, attempt to neatly define your approach to the problem and why you are stuck.
- If part of a problem depends on a previous part that you are unable to solve, explain the method for doing the current part, and, if possible, give the answer in terms of the quantities of the previous part that you are unable to obtain.
- Start each problem on a new page. Not only will this facilitate grading but also make it easier for you to jump back and forth between problems.
- If you get to the end of the problem and realize that your answer must be wrong (e.g. a negative probability), be sure to write “this must be wrong because …” so that I will know you recognized such a fact.
- Academic dishonesty **will** be dealt with **harshly** - the *minimum penalty* will be an “F” for the course.

1. Huffman Coding:

   [10] (a) Consider an independent and identically distributed (IID) sequence (X_i), where each X_i is drawn from the alphabet $\mathcal{X} = \{A, B\}$ and the probability mass function of each of the X_i is given by:

$$p_{X_i}(x) = \begin{cases} 0.7, & x = A \\ 0.2, & x = B \\ 0.1, & x = C \end{cases}$$

   You quickly use your calculator to find the entropy of this source is $H(X) = 1.16$ bits.

   Design a Huffman code that takes blocks of length 2 characters (i.e. $N = 2$) and find its rate (in output bits per input character). **Also** show that the rate falls between easily obtained upper and lower bounds based on the source entropy $H(X)$.

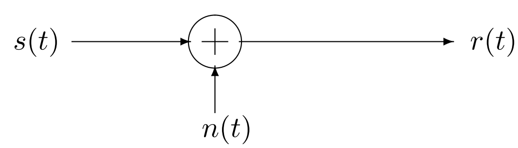
   [10] (b) We are considering the Huffman coding of a set of 8 input blocks. Suppose we order the blocks in decreasing order of probability, and suppose that no two blocks have the same probability (i.e the blocks have strictly decreasing probability as we move down the list). Indicate why each of the following codes is **not** a Huffman code, where the codewords are listed in the same order as the blocks; that is, the first bit string in the sequence is the code for the highest probability block, the second bit string in the sequence is for the second most probable block, etc.

   (i) $00, 01, 100, 101, 110, 111, 1110, 1111$

   (ii) $00, 01, 100, 1100, 1101, 101, 1110, 1111$

   (iii) $00, 01, 100, 1010, 1011, 1100, 1110, 1111$

2. Consider the waveform channel:



   where $n(t)$ is additive white Gaussian noise with power spectral density $\frac{N_0}{2}$, $r(t)$ is the received waveform, and $s(t) = s_i(t)$ when message m_i is to be sent during time $t \in (0, 2)$. Suppose there are $M = 2$ possible **equally likely** messages and the corresponding signals are:

$$s_1(t) = \begin{cases} 1, & 0 \le t \le 2 \\ 0, & \text{else} \end{cases}$$

$$s_2(t) = \begin{cases} 2 - t, & 0 \le t \le 2 \\ 0, & \text{else} \end{cases}$$

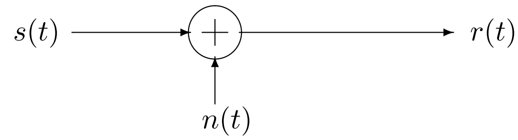
   [15] (a) Find an orthonormal basis $\{\phi_j(t) : j = 1, \dots, N\}$ with minimum N for these signals, and give the vector representation of each of the signals in this basis. **Go slowly and be careful with your algebra.**

   [5] (b) Specify the MAP receiver by:

   - Drawing a (simple) block diagram showing a method of obtaining r_j (the component of $r(t)$ along the basis function $\phi_j(t)$) from $r(t)$.
   - Drawing the decision regions in $\underline{r}$-space ($\underline{r} = (r_1, \dots, r_N)^T$), showing where each signal is chosen.

   [10] (c) Find the probability of symbol error $P(E)$ in terms of the average energy per symbol E_s and N_0. How much better (or worse) is this (in dB) than binary phase shift keying (BPSK)?

3. Consider the waveform channel:



where $n(t)$ is additive white Gaussian noise with power spectral density $\frac{N_0}{2}$, $r(t)$ is the received waveform, and $s(t) = s_i(t)$ when message m_i is to be sent during time $t \in (0, T_s)$. Suppose there are $M = 4$ possible **equally likely** messages and the corresponding signals are:

$$
\begin{aligned}
s_1(t) &= 0, & 0 \le t \le T_s \\
s_2(t) &= 2\cos(2\pi f_c t), & 0 \le t \le T_s \\
s_3(t) &= 2\sqrt{2}\cos\left(2\pi f_c t + \frac{\pi}{4}\right), & 0 \le t \le T_s \\
s_4(t) &= 2\sqrt{2}\cos\left(2\pi f_c t - \frac{\pi}{4}\right), & 0 \le t \le T_s
\end{aligned}
$$

where $f_c \gg T_s$ is the carrier frequency.

[10] (a) Find an orthonormal basis $\{\phi_j(t) : j = 1, \ldots, N\}$ with minimum N for these signals, and give the vector representation of each of the signals in this basis. (**Hint: You should be able to do this without doing Gram-Schmidt!**)

[5] (b) Specify the MAP receiver by:

- Drawing a (simple) block diagram showing a method of obtaining r_j (the component of $r(t)$ along the basis function $\phi_j(t)$) from $r(t)$.
- Drawing the decision regions in $\underline{r}$-space ($\underline{r} = (r_1, \ldots, r_N)^T$), showing where each signal is chosen.

[10] (c) Find the Union Bound to the probability of error in terms of the average signal energy E_s and N_0. **Then**, convert your answer to be in terms of the average energy per bit E_b and N_0.

[5] (d) For what SNRs Is your answer to part (c) close to the actual $P(E)$: low, moderate, or high? Why is this property convenient for the engineering of digital communication systems?

[5] (e) Focusing just on the term related to d_{min} (in terms of E_b) for both this signal set and QPSK, how much better or worse is this signal set than QPSK (in dB)?

4. Consider a channel where:

$$r = s + n$$

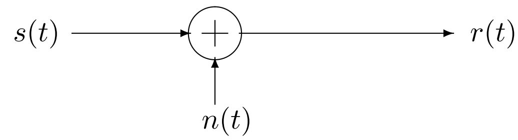
where s is the transmitted signal, which is equally likely to be $s_1 = 0$ or $s_2 = +6$. Suppose the noise has a Laplacian probability density function:

$$p_n(x) = \frac{1}{2} e^{-|x|}$$

[8] (a) Derive (don't guess!) the optimal decision regions to choose between s_1 and s_2.

[7] (b) Find the probability of symbol error $P(E)$ in terms of units of the signal space (no need to convert to E_s).

5. [645 only] Consider the waveform channel:



where $n(t)$ is additive white Gaussian noise with power spectral density $\frac{N_0}{2}$, $r(t)$ is the received waveform, and $s(t) = s_i(t)$ when message m_i is to be sent during time $t \in (0, 2)$. Suppose there are $M = 2$ possible **equally likely** messages and the corresponding signals are:

$$s_1(t) = \begin{cases} 1, & 0 \le t \le 2 \\ 0, & \text{else} \end{cases}$$

$$s_2(t) = \begin{cases} 2 - t, & 0 \le t \le 2 \\ 0, & \text{else} \end{cases}$$

Suppose that we choose (erroneously!) the orthogonal (but not orthonormal) “basis” functions:

$$\phi_1(t) = \begin{cases} 4, & 0 \le t \le 1 \\ 0, & \text{else} \end{cases}$$

$$\phi_2(t) = \begin{cases} 4, & 1 \le t \le 2 \\ 0, & \text{else} \end{cases}$$

and generate:

$$r_1 = \int_0^2 r(t)\phi_1(t)dt$$

and

$$r_2 = \int_0^2 r(t)\phi_2(t)dt$$

[7] (a) For the optimal processing of r_1 and r_2 to determine which message was sent, draw the decision regions in $\underline{r}$-space ($\underline{r} = (r_1, \ldots, r_N)^T$), showing where each signal is chosen.

[10] (b) Find the probability of symbol error $P(E)$ in terms of the average transmitted energy per symbol E_s and N_0.

[3] (c) How much better (or worse) is this (in dB) than binary phase shift keying (BPSK)?