ECE 564/645 - Digital Communication Systems (Spring 2013)

Midterm Exam #1

Monday, March 11th, 7:00-9:00pm, Marston 211

Overview

- The exam consists of four problems for 100 points. The points for each part of each problem are given in brackets - you should spend your **two hours** accordingly.
- The exam is closed book, but you are allowed **two page-sides** of notes. Calculators are **not** allowed. I will provide all necessary blank paper.

Testmanship

- **Full credit will be given only to fully justified answers.**
- Giving the steps along the way to the answer will not only earn full credit but also maximize the partial credit should you stumble or get stuck. If you get stuck, attempt to neatly define your approach to the problem and why you are stuck.
- If part of a problem depends on a previous part that you are unable to solve, explain the method for doing the current part, and, if possible, give the answer in terms of the quantities of the previous part that you are unable to obtain.
- Start each problem on a new page. Not only will this facilitate grading but also make it easier for you to jump back and forth between problems.
- If you get to the end of the problem and realize that your answer must be wrong (e.g. a negative probability), be sure to write "this must be wrong because ..." so that I will know you recognized such a fact.
- Academic dishonesty **will** be dealt with **harshly** - the *minimum penalty* will be an "F" for the course.

1. Short answer questions:

   [5] (a) What defines a communication system as **digital**?

   [5] (b) At what signal-to-noise ratios $\frac{E_s}{N_0}$ (small, moderate, or large) is the Union Bound to the symbol error probability of MAP receivers for the AWGN channel tight, where "tight" is defined as the bound being very close to the actual probability of error? No justification is required. *Briefly note why your answer makes the Union Bound useful.*

   [10] (c) Your boss wants to send 3 bits per symbol and is trying to decide whether to use 8-PSK, or an 8-QAM signal set with points:

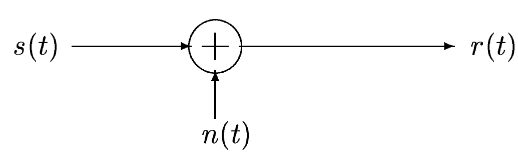
   $$(-3,-1),(-3,+1),(-1,-1),(-1,+1),(+1,-1),(+1,+1),(+3,-1),(+3,+1).$$

   He asks you to figure out which is better in energy efficiency at high signal-to-noise ratios (SNRs), and by how much? He needs an approximate answer in 10 minutes, so use a simple minimum distance argument to give an answer.

2. Consider the following on-off keying system for transmitting a bit b_0, **equally likely** to be 0 or 1, in $t \in (0,1)$. For $b_0 = 0$, we let $s(t) = 0$, and for $b_0 = 1$, we let $s(t) = \sqrt{2E_s}p(t)$, where

   $$p(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

   The signal $s(t)$ is transmitted across a channel modeled as the following:



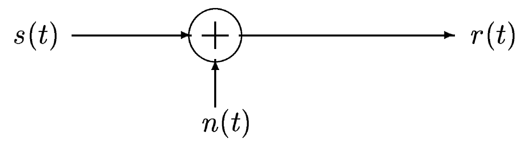
   where $n(t)$ is additive white Gaussian noise with power spectral density $\frac{N_0}{2}$ and $r(t)$ is the received waveform.

   [7] (a) Find the receiver for processing $r(t), t \in (0,1)$ to obtain an estimate for the transmitted bit that minimizes the probability of a bit error.

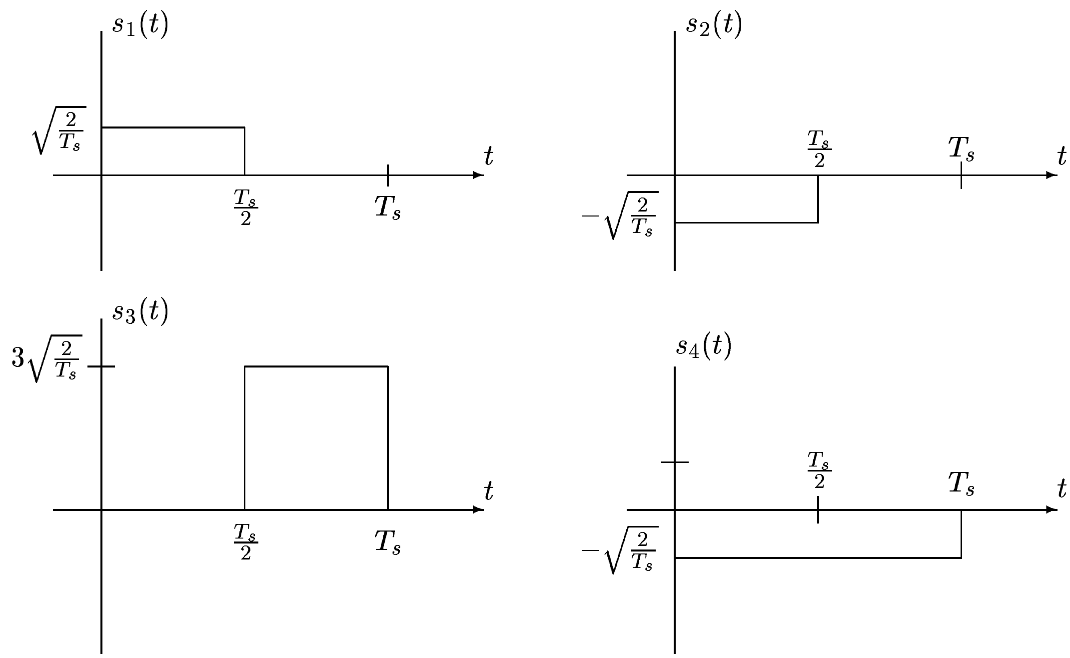
   [8] (b) Find the probability of a bit error in terms of the average energy per symbol E_s and N_0.

   [5] (c) How much better or worse (in dB of $\frac{E_s}{N_0}$) is this system than a binary phase-shift keyed (BPSK) system operating on an AWGN channel?

3. Consider the waveform channel:



where $n(t)$ is additive white Gaussian noise with power spectral density $\frac{N_0}{2}$, $r(t)$ is the received waveform, and $s(t) = s_i(t)$ when message m_i is to be sent during time $t \in (0, T_s)$. Suppose there are $M = 4$ possible **equally likely** messages and the corresponding signals are:



[10] (a) Find an orthonormal basis $\{\phi_j(t) : j = 1, \ldots, N\}$ with minimum N for these signals, and give the vector representation of each of the signals in this basis. (You don't have to explicitly use Gram-Schmidt, but then you must comment on how you *know* your basis minimizes N.)

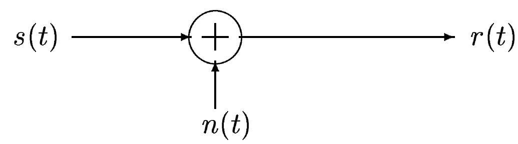
[5] (b) Specify the MAP receiver by:

- Drawing a (simple) block diagram showing a method of obtaining r_j (the component of $r(t)$ along the basis function $\phi_j(t)$) from $r(t)$.
- Drawing the decision regions in $\underline{r}$-space ($\underline{r} = (r_1, \ldots, r_N)^T$), showing where each signal is chosen. You do not have to give precise intercepts and such for the boundaries - just approximate the picture.

[10] (c) Give the Union bound on the symbol error probability of the MAP receiver. It is sufficient to express the bound as a function of the units of the vector space - you need *not* convert to average energy E_s.

[5] (d) Recall that the Union Bound for $P(E)$ is obtained by union bounding $P(E|s_i)$ for each i. Consider $P(E|s_3)$. Argue that one of the $M-1$ terms from the union bound on $P(E|s_3)$ can be removed and still yield an upper bound.

4. Consider the waveform channel:



where $n(t)$ is additive white Gaussian noise with power spectral density $\frac{N_0}{2}$, $r(t)$ is the received waveform, and $s(t) = s_i(t)$ when message m_i is to be sent during time $t \in (0, T_s)$. Suppose there are $M = 2$ possible **equally likely** messages and the corresponding signals are:

$$\begin{aligned} s_1(t) &= \sqrt{2P_c}\cos(2\pi f_c t), \quad 0 \le t \le T_s \\ s_2(t) &= -\sqrt{2P_c}\cos(2\pi f_c t), \quad 0 \le t \le T_s \end{aligned}$$

where P_c is the transmitted power and f_c is the carrier frequency.

[5] (a) Specify the optimal (MAP) processing of $r(t), t \in [0, T_s]$ for determining which message was sent. What is the probability of error of the MAP receiver in terms of $E_s = P_c T_s$ and N_0?

[10] (b) The **same exact processing from (a)** is employed but **unknown to the receiver,** the transmitted signals are really given by

$$\begin{aligned} s_1(t) &= \sqrt{2P_c}\cos(2\pi f_c t + \theta_\epsilon), \quad 0 \le t \le T_s \\ s_2(t) &= -\sqrt{2P_c}\cos(2\pi f_c t + \theta_\epsilon), \quad 0 \le t \le T_s \end{aligned}$$

where $0 \le \theta_\epsilon \le \frac{\pi}{2}$. Find the probability of error of this system as a function of θ_ϵ. (Assume $f_c \gg \frac{1}{T_s}$).

5. Consider an independent and identically distributed (IID) sequence (X_i), where each X_i is drawn from the alphabet $\mathcal{X} = \{A, B, C\}$ and the probability mass function of each of the X_i is given by:

$$p_{X_i}(x) = \begin{cases} 0.5, & x = A \\ 0.3, & x = B \\ 0.2, & x = C \end{cases}.$$

You quickly use your calculator to find the entropy of this source is $H(X) = 1.485$ bits.

[10] (a) Design a Huffman code that takes blocks of length 2 characters (i.e. $N = 2$) and find its rate (in output bits per input character). Show that the rate falls between easily obtained upper and lower bounds based on the source entropy $H(X)$.

[5] (b) Without finding the Huffman code (this would take far too much time!), find lower and upper bounds to the rate (in output bits per input character) of a Huffman code that takes symbols eight at a time (i.e. $N = 8$). Full credit goes to the tightest bounds.