ECE 603 - Probability and Random Processes, Fall 2016

Midterm Exam #1

October 19th, 7:00-9:00pm

Integrated Learning Center (ILC), Room S240

Overview

- The exam consists of five problems for 135 points. The points for each part of each problem are given in brackets - you should spend your **two hours** accordingly.
- The exam is closed book, but you are allowed **one page-side** of notes. Calculators are not allowed. I will provide all necessary blank paper.

Testmanship

- **Full credit will be given only to fully justified answers.**
- Giving the steps along the way to the answer will not only earn full credit but also maximize the partial credit should you stumble or get stuck. If you get stuck, attempt to neatly define your approach to the problem and why you are stuck.
- If part of a problem depends on a previous part that you are unable to solve, explain the method for doing the current part, and, if possible, give the answer in terms of the quantities of the previous part that you are unable to obtain.
- Start each problem on a new page. Not only will this facilitate grading but also make it easier for you to jump back and forth between problems.
- If you get to the end of the problem and realize that your answer must be wrong, be sure to write "this must be wrong because . . . " so that I will know you recognized such a fact.
- Academic dishonesty will be dealt with harshly - the *minimum penalty* will be an "F" for the course.

1. Consider a random variable X with cumulative density function $F_X(x)$ given by:

$$F_X(x) = \begin{cases} 0, & x < -2 \\ -\frac{1}{24}x^2 + \frac{1}{6}x + \frac{5}{6}, & -2 \le x < 2 \\ 1, & x \ge 2 \end{cases}$$

[5] (a) Find $P(X > 1)$.

[5] (b) Find $P(-2 \le X \le 1)$.

[5] (c) Find $P(X^2 + 8\cos(X) < -9)$.

[8] (d) Find $f_X(x)$, the probability density function (pdf) of X.

[7] (e) Find $E[X^2]$.

[10] (f) Suppose $Y = |X|$. Find $f_Y(y)$.

[10] (g) Consider the mapping $Z : \mathcal{R} \to \{\text{Ballerina, Carpenter, Dentist}\}$ defined by:

$$Z(X) = \begin{cases} \text{Ballerina}, & X \le -2 \\ \text{Carpenter}, & -2 < X \le 1 \\ \text{Dentist}, & X > 1 \end{cases}$$

Find a probability space $(S, \mathcal{A}, P)$ for the outcome of this mapping. Since the space is small, write out each set $A \in \mathcal{A}$ and give its probability $P(A)$. (*Note: You can use "B", "C", and "D" as shorthand for "Ballerina", "Carpenter", and "Dentist", respectively.*)

2. [15] Suppose I have a cube, 2 inches on each side, and inside that cube I have a sphere of radius 1 (which, of course, just barely fits). I pick a point at random in that cube and form the random variable X in this manner:

   - If the point is within the sphere, I let X be the distance of that point from the center of the sphere.
   - If the point does not lie within the sphere, I let $X = 1$.

   You may (or may not) find the following formulas helpful: (i) volume of a sphere = $\frac{4}{3}\pi r^3$; (ii) surface area of a sphere = $4\pi r^2$.

   Find $f_X(x)$, the probability density function of X.

3. [15] Suppose I (attempt to) define an experiment where I draw a number "equally likely" from all of the rational numbers in $[0, 1]$. Show that this leads to an inconsistent probability space. (*Hint: Start off by finding what must be the probability of any singleton for such an experiment. Then, see if the axioms of probability can hold.)*

4. I roll an *unfair* 6-sided die that shows a "1" with probability $2/7$, and a "2", "3", "4", "5" or "6", each with probability $1/7$.

   [10] (a) Find a probability space $(S, \mathcal{A}, P)$ with the largest possible $\mathcal{A}$ for this experiment. You do not have to write out all of the elements of $\mathcal{A}$, but do be sure to specify $P(\cdot)$ so that I could find the probability of any $A \in \mathcal{A}$ from such.

   [5] (b) Find the smallest (i.e. that with the fewest elements) σ-algebra that contains ϕ, S, $\{1\}$ and $\{1, 2\}$.

   [10] (c) Suppose somebody takes this unfair 6-sided die and puts it into a bag with $K - 1$ fair 6-sided dice. I pull one die at random from the bag (i.e. each die is drawn with probability $\frac{1}{K}$), roll that one die 20 times (i.e. the die is drawn *once* and rolled 20 times), and I observe "1" 8 times.

   - Write an expression for the probability that the die I pulled from the bag is the unfair die.
   - Is your answer larger or smaller than $\frac{1}{K}$? (*A short justification in words is okay for this part.)*

5. I am taking (yet another) fictitious exam. This exam starts with an essay question Q_1, for which my number of points S_1 is a random variable drawn from the pmf:

$$P(S_1 = x_j) = \begin{cases} 1/8, & x_j = 1, 2, 3, 4 \\ 1/2, & x_j = 5 \\ 0, & \text{else} \end{cases}$$

   The exam continues with 5 other questions $Q_2, Q_3, \ldots, Q_6$, for each of which I either get zero points ($S_i = 0$) or one point ($S_i = 1$). For each of these 5 questions, my probability of getting the answer correct (and thus $S_i = 1$) depends on how well I did on Q_1;

   - If I got $S_1 = 5$ points (perfect) on Q_1, then $P(S_i = 1) = 0.8$ for $i = 2, 3, \ldots, 6$ and, conditioned on the result for Q_1, the outcome for each question is independent of the other questions.
   - If I got $S_1 = j$, $j = 1, 2, 3$ or 4 points on Q_1, then $P(S_i = 1) = 0.5$ for $i = 2, 3, \ldots, 6$ and, conditioned on the result for Q_1, the outcome is independent of the other questions.

   [5] (a) What is the probability that I get Q_2 correct?

   [5] (b) Given that I get Q_2 correct, what is the probability that $S_1 = 2$.

   [5] (c) Given that I get Q_2 correct, what is the probability that I get Q_3 correct?

   [7] (d) What is my expected exam score $E[\sum_{i=1}^{6} S_i]$ (i.e. total of points from all 6 questions)?

   [8] (e) Let A be the event that I score ≥ 8 points (and pass the exam). Find an expression for $P(A)$; you do not have to evaluate it, but do not include terms that are zero. (*Hint: If you are writing lots of set expressions, you are going about it the hard way.*)