ECE 603 - Probability and Random Processes, Fall 2015

Midterm Exam #1

October 21st, 7:00-9:00pm

Ag. Engineering, Room 119

Overview

- The exam consists of five problems for 130 points. The points for each part of each problem are given in brackets - you should spend your **two hours** accordingly.
- The exam is closed book, but you are allowed **one page-side** of notes. Calculators are not allowed. I will provide all necessary blank paper.

Testmanship

- **Full credit will be given only to fully justified answers.**
- Giving the steps along the way to the answer will not only earn full credit but also maximize the partial credit should you stumble or get stuck. If you get stuck, attempt to neatly define your approach to the problem and why you are stuck.
- If part of a problem depends on a previous part that you are unable to solve, explain the method for doing the current part, and, if possible, give the answer in terms of the quantities of the previous part that you are unable to obtain.
- Start each problem on a new page. Not only will this facilitate grading but also make it easier for you to jump back and forth between problems.
- If you get to the end of the problem and realize that your answer must be wrong, be sure to write "this must be wrong because ..." so that I will know you recognized such a fact.
- Academic dishonesty will be dealt with harshly - the *minimum penalty* will be an "F" for the course.

Hint: You may find the following fact useful as you solve this exam:

$$P\left(\bigcap_{n=1}^{\infty}\left(x-\frac{1}{n},x+\frac{1}{n}\right)\right)=\lim_{n\to\infty}P\left(\left(x-\frac{1}{n},x+\frac{1}{n}\right)\right)$$

1. [10] An irrational number x is an *algebraic irrational* number if there exists an n such that the number x is a zero of a polynomial of degree n with integer coefficients. Consider the set A of all algebraic irrational numbers. Is the set A countable? (*Big Hint: First consider the number of such polynomials for some fixed degree n (e.g. $n = 3$) and the corresponding possible number of zeroes; then, extend to consider the collection of such over all n.*)

2. Suppose you take an exam with three questions, which have the following point values:

   Question 1: 20 points
   Question 2: 10 points
   Question 3: 10 points

   On this fictitious exam, there is no partial credit for a given question; hence, you either get the question correct (full points), or you get it wrong (no points).

   The probability you get Question 1 correct is 0.90. Unfortunately, Question 2 is dependent on Question 1: if you get Question 1 correct, there is a probability of 0.9 that you will also get Question 2 correct. However, if you get Question 1 wrong, there is a probability of 0.3 that you will get Question 2 correct. Question 3 is independent of Questions 1 and 2, and you get Question 3 correct with probability 0.5.

   Let Q_i be the event you get Question i correct.

   [5] (a) Find the probability that you get Question 2 correct; in other words, find $P(Q_2)$.

   [5] (b) Given that you got Question 2 correct, find the probability that you got Question 1 correct.

   [5] (c) Given that you got Question 3 correct, find the probability that you got Question 1 correct.

   [10] (d) Suppose that you pass the exam if you score greater than or equal to 30 points. Let A be the event that you pass the exam.

   - Write A in terms of Q_1, Q_2, and Q_3.
   - Find $P(A)$.

3. I have a die that, when rolled, displays one of three equally likely colors: red, green, and blue. Die rolls are independent from one another.

   [8] (a) I roll the die three times. What is the probability that I see one of each color?

   [5] (b) I roll the die 20 times. What is the probability I see red exactly 8 times?

   [7] (c) I roll the die n times. What is the probability (as a function of n) that I see at least one of each color?

4. Consider the probability space $(S, \mathcal{A}, P)$, with $S = [0, 1]$ and $\mathcal{A} = \mathcal{B}$ (restricted to $[0, 1]$,of course), and $P(\cdot)$ defined as follows:

$$P((a,b)) = \begin{cases} c \cdot ((b-a) - \frac{1}{2}(b^2 - a^2)), & 0 \leq a < b \leq \frac{1}{2} \\ c \cdot ((b-a) - \frac{1}{2}(b^2 - a^2)) & \frac{1}{2} \leq a < b \leq 1 \\ \frac{1}{2} + c \cdot ((b-a) - \frac{1}{2}(b^2 - a^2)), & a < \frac{1}{2} < b \leq 1 \end{cases}$$

[5] (a) Find c.

[15] (b) Let ω be the outcome of the experiment. Since ω is a number, it can be treated like a random variable, with cumulative distribution function $F_\omega(x) = P(\omega \leq x)$ and probability density function $f_\omega(x) = \frac{d}{dx}F_\omega(x)$.

- Find the probability density function $f_\omega(x)$.
- Find $E[\omega^2]$.

[10] (c) Let ω be the outcome of the experiment. Consider the mapping $Y : S \to \{\text{Alice, Bob, Carol, Dan}\}$ defined by:

$$Y(\omega) = \begin{cases} \text{Alice}, & 0 \leq \omega < 0.3 \\ \text{Bob}, & 0.3 \leq \omega < 0.6 \\ \text{Carol}, & 0.6 \leq \omega < 0.9 \\ \text{Dan}, & 0.9 \leq \omega \leq 1 \end{cases}$$

Find the probability space $(S_Y, \mathcal{Y}, P_Y)$ for the experiment with outcome Y. Please be *explicit* as this is a very small space! (You can use the abbreviations A, B, C, and D for the names).

[10] (d) Let ω be the outcome of the experiment. Suppose, I define the random variable X by:

$$X(\omega) = \begin{cases} \omega^2, & 0 < \omega \leq \frac{1}{2} \\ \frac{1}{2}, & \omega > \frac{1}{2} \end{cases}$$

Find the probability density function $f_X(x)$ of X.

[5] (e) Let ω be the outcome of the experiment. Suppose I define the random variable Z by $Z = 2e^{-\omega^3} + e^{-\omega^2} + e^{-\omega} + \omega$. Find $P(Z > 6)$.

Be sure to see Problem 5 on the next page!

5. Each day, I receive a shipment of 3 parts from a single warehouse, equally likely to be warehouse W_1 or warehouse W_2.

The number of defective parts D_1 given it comes from warehouse W_1 obeys the cumulative distribution function (CDF) given by:

$$F_{D_1}(x) = \begin{cases} 0, & x < 0 \\ 1/2, & 0 \leq x < 1 \\ 8/10, & 1 \leq x < 2 \\ 9/10, & 2 \leq x < 3 \\ 1, & 3 \leq x \end{cases}$$

The number of defective parts D_2 given it comes from warehouse W_2 obeys the cumulative distribution function (CDF) given by:

$$F_{D_2}(x) = \begin{cases} 0, & x < 0 \\ 1/10, & 0 \leq x < 1 \\ 4/10, & 1 \leq x < 2 \\ 7/10, & 2 \leq x < 3 \\ 1, & 3 \leq x \end{cases}$$

Given the warehouse, the number of defective parts from one day to another is independent. And the number of defective parts from one warehouse is independent of the number of parts that are defective from the other warehouse, meaning the event $\{D_1 = x\}$ is independent of the event $\{D_2 = y\}$ for any x and y.

[5] (a) Given the choice, from which warehouse would you choose to receive parts? (and why?)

[5] (b) On a given day, I receive a shipment of 3 parts from a single warehouse, equally likely to be warehouse W_1 or warehouse W_2. What is the probability that I receive more than 2 defective parts?

[5] (c) On a given day, I receive a shipment of 3 parts from a single warehouse, equally likely to be warehouse W_1 or warehouse W_2. A truck arrives on that day and it has two defective parts. What is the probability it came from warehouse W_2?

[5] (d) Suppose that, once we choose a warehouse, we stick with it forever (i.e. the truck comes from the same warehouse every day). My boss is very interested in promoting "error-free days", which are days we receive a shipment with no defective parts. What is the probability we have exactly 7 "error free days" in 10 days? (You can just write the expression, of course.)

[10] (e) *(This part is independent of part (d).)* The shipping company offers me (at a price) the following option: rather than receiving parts from a randomly chosen truck, I can look at the number of defects on each of the two trucks and then take the shipment with the fewer number of defects. For a given day, let the number of defects for this strategy be the random variable D, which is the minimum of D_1 and D_2 for that day. Find the probability mass function for D.