ECE 603 - Probability and Random Processes, Fall 2010

Midterm Exam #1

October 20th, 6:30-8:30pm

Overview

- The exam consists of five problems for 130 points. The points for each part of each problem are given in brackets - you should spend your **two hours** accordingly.
- The exam is closed book, but you are allowed **one page-side** of notes. Calculators are not allowed. I will provide all necessary blank paper.

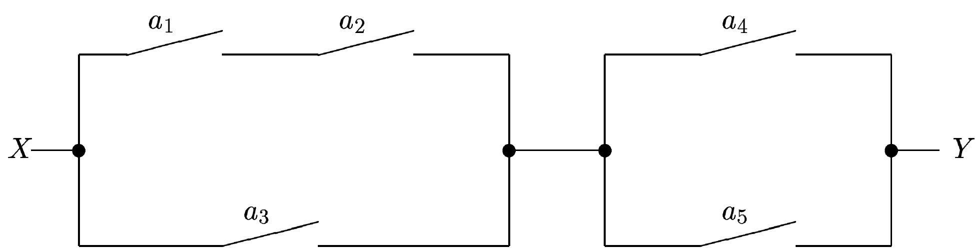
Testmanship

- **Full credit will be given only to fully justified answers.**
- Giving the steps along the way to the answer will not only earn full credit but also maximize the partial credit should you stumble or get stuck. If you get stuck, attempt to neatly define your approach to the problem and why you are stuck.
- If part of a problem depends on a previous part that you are unable to solve, explain the method for doing the current part, and, if possible, give the answer in terms of the quantities of the previous part that you are unable to obtain.
- Start each problem on a new page. Not only will this facilitate grading but also make it easier for you to jump back and forth between problems.
- If you get to the end of the problem and realize that your answer must be wrong, be sure to write "this must be wrong because . . . " so that I will know you recognized such a fact.
- Academic dishonesty will be dealt with harshly - the *minimum penalty* will be an "F" for the course.

Hint: You may find the following fact useful as you solve this exam:

$$P\left(\bigcap_{n=1}^{\infty}(x-\frac{1}{n},x+\frac{1}{n})\right)=\lim_{n\to\infty}P((x-\frac{1}{n},x+\frac{1}{n}))$$

1. Consider the network:



and let $A_i, i = 1,2,3,4,5$ denote the probability that switch a_i is closed (i.e. that packets can flow through that switch). Now, suppose that the events A_i have the following properties:

- The status of switches a_4 and a_5 are independent of the status of switches a_1, a_2 and a_3.
- A_1 and A_2 are independent.
- $P(A_1) = P(A_2) = 0.5$.
- $P(A_3|(A_1 \bigcap A_2)) = 0.4$.
- $P(A_3|(\overline{A}_1 \bigcup \overline{A}_2)) = \frac{1}{3}$.
- A_4 and A_5 form a partition.

Let C be the event that there is a connection between X and Y.

[7] (a) Find $P(A_3)$.

[8] (b) Write an expression for C in terms of unions and intersections of the A_i's (and their complements), and find $P(C)$.

[5] (c) Given that C occurs, find the probability that a_3 is closed.

[10] (d) Suppose that $P(A_3|(A_1 \bigcap A_2))$ and $P(A_3|(\overline{A}_1 \bigcup \overline{A}_2))$ can be re-designed, *but* $P(A_3)$ must remain the same as in (a).

- How would you specify $P(A_3|(A_1 \bigcap A_2))$ and $P(A_3|(\overline{A}_1 \bigcup \overline{A}_2))$ such that $P(C)$ is maximized?
- How would you specify $P(A_3|(A_1 \bigcap A_2)$ and $P(A_3|(\overline{A}_1 \bigcup \overline{A}_2)$ such that $P(C)$ is minimized?

2. Suppose that the probability space $(\Omega, \mathcal{A}, P)$ is defined by $\Omega = [0,1]$, $\mathcal{A} = \mathcal{B}$ (restricted to $[0,1]$, of course), and $P((a,b)) = (b^2 - a^2)$.

[10] (a) Define the random variable $X : \Omega \to \mathcal{R}$ by:

$$X(\omega) = \begin{cases} \omega, & 0 < \omega < 1/2 \\ 2/3, & 1/2 \leq \omega < 1 \end{cases}$$

- Find the probability density function $f_X(x)$ for X.
- What is the probability X falls in the interval $(\frac{2}{3}, 1)$?

[10] (b) Consider the mapping (**not a random variable!**) $Y : \Omega \to$ {Apple, Banana, Cardinal, Dove} given by

$$Y(\omega) = \begin{cases} \text{Apple}, & 0 \leq \omega < 1/8 \\ \text{Banana}, & 1/8 \leq \omega < 1/4 \\ \text{Cardinal}, & 1/4 \leq \omega < 1/2 \\ \text{Dove}, & 1/2 \leq \omega \leq 1 \end{cases}$$

- Find the induced probability space for Y (feel free to use the first letters "A", "B", "C", and "D" of Apple, Banana, Cardinal, and Dove, respectively, to save you space)
- Define the events $F =$ {Apple, Banana} and $R =$ {Apple, Cardinal}. Find $P(R|F)$.

3. Suppose that I conduct the following experiment. I flip a coin continually, and I write down the result of the flips to get an outcome ω. For example, a possible ω might look like:

$$\omega = (\text{Tail, Tail, Tail, Head, Head}, \ldots)$$

[10] (a) Consider the sample space S of all possible outcomes ω. Is S countable or uncountable?

[10] (b) Construct a rich (i.e. non-trivial) probability space $(S, \mathcal{A}, P)$ for this experiment. You should end up with an $\mathcal{A}$ with an uncountable number of elements that should allow one to answer any question of interest.

4. Let X be a random variable with probability density function:

$$f_X(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{else} \end{cases}$$

and let Z be a random variable with probability density function

$$f_Z(z) = \frac{1}{3}\delta(z-1) + \frac{2}{3}\delta(z-2)$$

Answer the following six parts independently. **In each case, the answer and a single line of justification is sufficient.**

[5] (a) Let $Y = X + 2$. Find $f_Y(y)$.

[5] (b) Let $Y = -X$. Find $f_Y(y)$.

[5] (c) Let $Y = g(X)$, where

$$g(x) = \begin{cases} 5, & -2 < x < 2 \\ 0, & \text{else} \end{cases}$$

Find $f_Y(y)$.

[7] (d) Find $E[Z]$ and $E[Z^2]$.

[8] (e) Let $Y = 4Z^2 + 6Z + 8$. Find $f_Y(y)$.

[5] (f) Suppose that I run 10 trials, each resulting in an independent observation of a random variable with probability density function $f_Z(z)$. What is the probability that the sum of the outcomes exceeds 15?

5. The random variable X has probability density function:

$$f_X(x) = c\, e^{(x - \frac{x^2}{6})}, \qquad -\infty < x < \infty$$

which is shown below (**Be sure to look at the plot!**).

[10] (a) Find c.

If you cannot get part (a), proceed now do do parts (b) through (d) as if X is Gaussian with mean 3 and variance 9. Make sure to note on your paper that you are doing such.

[5] (b) Find $E[X^2]$. (*Hint: You do not have to do any integrals - look at your equation for finding the variance of a random variable.*)

[5] (c) Find $P(X > 6)$.

[5] (d) Find $E[5X^2 + 4X + 2]$.

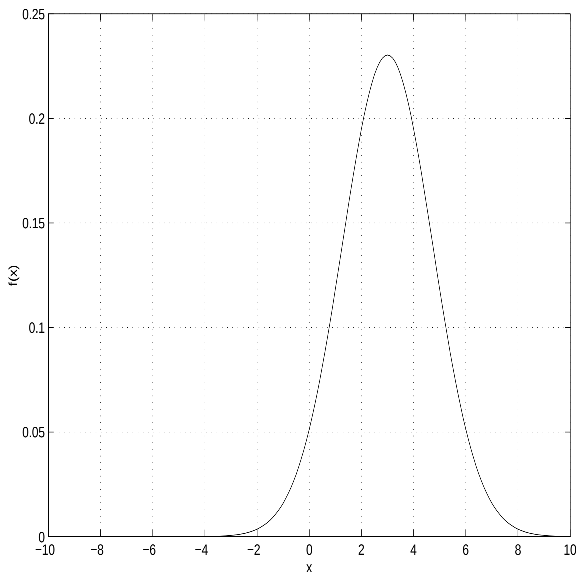


Figure 1: Probability density function of X for Problem 5.