Exam Notes:

- The exam consists of 4 problems for 100 points. The points for each part of each problem are given - you should spend your two hours accordingly.

- The exam is closed book, but you are allowed a one-sided page of notes. Calculators are not allowed. Space for answers is provided in the exam.

- Full credit will be given only to fully justified answers.

- Giving the steps along the way to the answer will not only earn full credit but also maximize the partial credit should you stumble or get stuck. If you get stuck, attempt to neatly define your approach to the problem and explain why you are stuck. Also, if you get an answer that you know must be wrong but you can’t find your mistake, explain how you know the answer is wrong.

- If part of a problem depends on a previous part that you are unable to solve, explain the method for doing the current part, and, if possible, give the answer in terms of the quantities of the previous part that you are unable to obtain.

- You should carry out calculations as far as you can. It’s ok if you are left with some terms whose numerical values you can’t find without a calculator (such as \( \left( \frac{10}{3} \right) \) or \( e^{1/2} \)), but you should complete simple calculations (e.g., a calculation like \( \frac{1}{3} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2} \)).
1. (30%) Suppose that events $A$, $B$, and $C$ form a partition of some sample space $S$, with $P(A) = \frac{1}{6}$ and $P(B) = \frac{1}{3}$. Say that we also have two other events $D$ and $E$. We know that:

- $P(D) = P(E) = \frac{1}{3}$;
- $P(D|A) = P(D|B) = \frac{2}{3}$;
- $P(D \cap E|A) = \frac{1}{3}$;
- and $P(D \cap E|B) = \frac{1}{6}$.

a. (5%) Find $P(C)$.
b. (5%) Find $P(D \cap C)$.
c. (5%) Find $P(D \cap E)$.
d. (5%) Find $P(A \cap E|D)$.
e. (5%) Find $P(A^c \cap (D \cap E)^c)$.
f. (5%) Find $P(D \cup A^c)$.
2. **(25%)** Suppose that we have a text generator that randomly chooses a sequence of characters $c_1, c_2, c_3, \ldots$ from the English alphabet. (Note that in the English alphabet, five letters (A, E, I, O, U) are vowels and the remaining 21 letters are consonants.)

a. **(5%)** Suppose first that the characters are chosen independently, that each letter of the alphabet is equally likely to be chosen for each character, and that duplicate letters are allowed among the chosen characters. Find the probability that there is exactly one vowel in the first three characters that are chosen.

b. **(5%)** Under the same conditions as part (a), find the expected number of vowels in the first three characters that are chosen.

c. **(5%)** Now we modify the text generator so that duplicate letters are **not** allowed (for example, if the generator chooses the letter B as the first character, then the next character is chosen from the set of letters \{A, C, D, \ldots Z\}). Find the probability that there is exactly one vowel in the first three characters that are chosen under these conditions.

d. **(5%)** Under the same conditions as part (c), find the probability that the first and third characters that are chosen are vowels.

e. **(5%)** We make one more modification to the text generator: The first character is drawn at random from the entire alphabet. If the first character that is chosen is a consonant, then the second character is drawn at random from the set of vowels. If the first character is a vowel, then the second character is drawn at random from the set of the other 25 letters. Find the probability that the second character is an A.
3. **(25%)** Suppose that we have a digital information source that randomly generates a pair of bits. Define the events $A = \{\text{first bit is a 1}\}$ and $B = \{\text{second bit is a 1}\}$. We know the following probabilities:

\[
P(A) = \frac{1}{3} \quad P(B) = \frac{2}{3} \quad P(B^c | A^c) = \frac{1}{3}.
\]

Define the random variable $X$ to be the number of 1’s in the generated bit pair.

a. **(5%)** Find the PMF $P_X(x)$.

b. **(5%)** Find $E[X]$ and $\text{Var}(X)$.

c. **(5%)** Find $P(X = 2 | B)$.

d. **(5%)** Now suppose that we observe two consecutive bit pairs generated by the source (that is, a total of four bits). Assume that the bit pairs are generated independently, and that the bits within each pair have the probabilities given at the beginning of this problem. Let $Y$ denote the total number of 1s in the four bits. Find $E[Y]$ and $\text{Var}(Y)$.

e. **(5%)** With $Y$ as defined in part (d), find $P(Y = 2)$. 
4. (20%) Suppose that a discrete random variable $X$ has the CDF shown below:

Also define another random variable $Y$ by $Y = X^2$.

a. (5%) Find the PMF $P_X(x)$.

b. (5%) Find the PMF $P_Y(y)$.

c. (5%) Find $P(Y > 1|X > 0)$.

d. (5%) Let $a$ and $b$ be constants, and define the random variable $Z = aY + b$ (where $Y$ is the random variable defined above). Find values of $a$ and $b$ so that $E[Z] = 0$ and $Var(Z) = 1$. 