Midterm #1 Solutions

ECE 603 | Fall, 2016

1) (a)

$$P(X>1) = 1 - P(X \le 1)$$
$$= 1 - F_X(1)$$
$$= 1 - \left(-\tfrac{1}{24} + \tfrac{1}{6} + \tfrac{5}{6}\right)$$
$$= \tfrac{1}{24}$$

(b) $P(-2 \le X \le 1) = P(X=-2) + P(-2 < X \le 1)$

$$= F_X(-2) - F_X(-2^-) + F_X(1) - F_X(-2)$$
$$= F_X(1) - F_X(-2^-)$$
$$= F_X(1) = \tfrac{23}{24}$$

(c) Note: ① $X^2 \ge 0$

② $8\cos X \ge -8$

Thus, $P(X^2 + 8\cos X < -9) = 0$

(d) $f_X(x) = \frac{d}{dx} F_X(x)$

Note there is a jump of size $F_X(-2) - F_X(-2^-)$ at $x=-2$;
$F_X(-2) - F_X(-2^-) = F_X(-2) = -\frac{1}{24}(-2)^2 + \frac{1}{6}\cdot -2 + \frac{5}{6} = -\frac{1}{6} - \frac{1}{3} + \frac{5}{6} = \frac{1}{3}$
The rest is differentiable; thus,

$$f_X(x) = \tfrac{1}{3}\,\delta(x+2) + \begin{cases} -\frac{1}{12}x + \frac{1}{6}, & -2 < x \le 2 \\ 0, & \text{else} \end{cases}$$

(e)

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_{-\infty}^{\infty} x^2 \cdot \tfrac{1}{3}\,\delta(x+2)\,dx + \int_{-2}^{2} \left(-\tfrac{1}{12}x^3 + \tfrac{1}{6}x^2\right) dx$$

$$= \tfrac{1}{3}\cdot(-2)^2 + -\tfrac{1}{48}x^4\Big|_{-2}^{2} + \tfrac{1}{18}x^3\Big|_{-2}^{2}$$

$$= \tfrac{4}{3} + \cancel{\left(-\tfrac{1}{3} + \tfrac{1}{3}\right)} + \tfrac{8}{18} + \tfrac{8}{18}$$

$$= \tfrac{12}{9} + \tfrac{8}{9} = \tfrac{20}{9}$$

(f)

$$F_Y(y) = P(Y \le y)$$

$$= P(|X| \le y)$$

$$= P(-y \le X \le y)$$

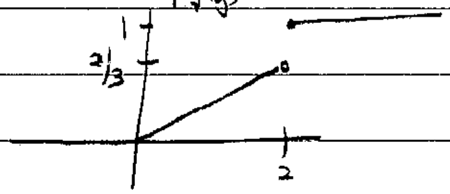
$0 \le y < 2$

$$= \int_{-y}^{y} \left(-\tfrac{1}{12}x + \tfrac{1}{6}\right) dx$$

$$= \left(-\tfrac{1}{24}x^2 + \tfrac{1}{6}x\right)\Big|_{-y}^{y}$$

$$= \left(-\tfrac{1}{24}y^2 + \tfrac{1}{6}y\right) - \left(-\tfrac{1}{24}y^2 - \tfrac{1}{6}y\right)$$

$$= \tfrac{1}{3}y$$



and

$$f_Y(y) = \tfrac{d}{dy} F_Y(y)$$

$$= \tfrac{1}{3}\,\delta(y-2) + \begin{cases} \tfrac{1}{3}, & 0 \le y \le 2 \\ 0, & \text{else} \end{cases}$$

Note: This integrates to 1!

g)

$S = \{B, C, D\}$

$\mathcal{A} = \mathcal{P}_S = \{\phi, \{B\}, \{C\}, \{D\}, \{B,C\}, \{B,D\}, \{C,D\}, S\}$

$P(A) = \sum_{x_i \in A} P(\{x_i\})$

$P(\{B\}) = P(X \leq -2) = 1/3$

$P(\{C\}) = P(-2 < X \leq -1) = 15/24 = 5/8$ (see (b))

$P(\{D\}) = 1/24$

$P(\{B, C\}) = 23/24$

$P(\{B, D\}) = 9/24 = 3/8$

$P(\{C, D\}) = 2/3$

$P(\phi) = 0$

$P(S) = 1$

2)

$$F_X(x) = P(X \leq x)$$

$$x < 1 \longrightarrow = \frac{4/3\, \pi \cdot x^3}{2^3}$$

$$= \pi x^3/6$$

Thus, $F_X(x) = \begin{cases} 0, & x < 0 \\ \pi x^3/6, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$

Note jump at $x = 1$ of $1 - \pi/6$!

$$\Rightarrow f_X(x) = d/dx\, F_X(x) = (1 - \pi/6)\,\delta(x-1) + \begin{cases} \pi x^2/2, & 0 \leq x < 1 \\ 0, & \text{else} \end{cases}$$

3) There are an infinite number of rationals.
I claim $P(X = x) = 0$, for any singleton x.
Suppose not. Then $P(X = x) = \varepsilon > 0$ for all $x \in \mathbb{Q}$.
But there exists at least $\lceil 2/\varepsilon \rceil$ rationals,
implying (by the third axiom applied on $N = \lceil 2/\varepsilon \rceil$ sets)
that $P(S) > 1$. Hence $P(X = x) = 0$ for any singleton x.
But $\mathbb{Q}$ is countable; thus,

$$1 \overset{\text{2nd axiom}}{=} P(S) = P(\mathbb{Q}) \overset{\text{3rd axiom, countable set } \mathbb{Q}}{=} \sum_{x_i \in \mathbb{Q}} P(\{x_i\}) = \sum_{x_i \in \mathbb{Q}} 0 = 0$$

inconsistent with axioms

4)

(a) $S = \{1,2,3,4,5,6\}$

$\mathcal{A} = \mathcal{P}_S$

For $A \in \mathcal{A}$, $P(A) = \sum_{x_i \in A} P(\{x_i\})$

where $P(\{1\}) = 2/7 \quad P(\{2\}) = P(\{3\}) = P(\{4\}) = P(\{5\}) = P(\{6\}) = 1/7$

(b)

$\{\emptyset, \{1\}, \{1,2\}, \{2\}, \{3,4,5,6\}, \{2,3,4,5,6\}, \{1,3,4,5,6\}, S\}$

(c)

Bayes

• $P(\text{unfair} \mid 8 \text{ "1"s}) = \dfrac{P(8 \text{ "1"s} \mid \text{unfair})\, P(\text{unfair})}{P(8 \text{ "1"s})}$

$$= \frac{\binom{20}{8}(2/7)^8(5/7)^{12}}{1/k\left(\binom{20}{8}(2/7)^8(5/7)^{12} + \underbrace{\binom{20}{8}(1/6)^8(5/6)^{12} + \cdots}_{k-1 \text{ times}}\right)}$$

$$= \frac{\binom{20}{8}(2/7)^8(5/7)^{12}}{\frac{1}{k}\left(\binom{20}{8}(2/7)^8(5/7)^{12} + (k-1)\binom{20}{8}(1/6)^8(5/6)^{12}\right)}$$

• Larger

Lots of 1's makes unfair die more likely.

5) (a)

$$P(Q_2) = P(Q_2 \mid S_1 = 5)\,P(S_1 = 5) + P(Q_2 \mid S_1 < 5)\,P(S_1 < 5)$$

$$= 0.8 \cdot \tfrac{1}{2} + 0.5 \cdot \tfrac{1}{2}$$

$$= 0.65$$

(b) $$P(S_1 = 2 \mid Q_2) \underset{\text{Bayes}}{=} \frac{P(Q_2 \mid S_1 = 2)\,P(S_1 = 2)}{P(Q_2)} = \frac{0.5 \cdot 1/8}{0.65} = \frac{1/16}{13/20} = 5/52$$

(c) $$P(Q_3 \mid Q_2) = \frac{P(Q_2 \cap Q_3)}{P(Q_2)}$$

$$= \frac{P(Q_2 \cap Q_3 \mid S_1 = 5)\,P(S_1 = 5) + P(Q_2 \cap Q_3 \mid S_1 < 5)\,P(S_1 < 5)}{P(Q_2)}$$

$$= \frac{0.64 \cdot \tfrac{1}{2} + 0.25 \cdot \tfrac{1}{2}}{0.65}$$

$$= \frac{0.32 + 0.125}{0.65} = 0.445/0.65$$

(d) $$E[\text{Total}] = E\Big[\sum_{i=1}^{6} S_i\Big]$$

$$= \sum_{i=1}^{6} E[S_i]$$

$$E[S_1] = \tfrac{1}{2} \cdot 5 + \tfrac{1}{8} \cdot 1 + \tfrac{1}{8} \cdot 2 + \tfrac{1}{8} \cdot 3 + \tfrac{1}{8} \cdot 4$$

$$= 5/2 + 10/8 = 15/4$$

$i = 2, 3, 4, 5, 6$

$$E[S_i] = 1 \cdot 0.65 + 0 \cdot 0.35 = 0.65$$

$$E[\text{total}] = 15/4 + 5 \cdot 0.65 = 7$$

(c)

Do Law of Total Probability, with partition the outcome of S_1

$$P(A) = \sum_{i=1}^{5} P(A \mid S_1 = i)\, P(S_1 = i)$$

Note that I cannot get ≥ 8 if $S_1 = 1$ or $S_1 = 2$.

$$P(A \mid S_1 = 3) = 0.5^5$$

$$P(A \mid S_1 = 4) = \binom{5}{4} 0.5^4\, 0.5^1 + 0.5^5$$

$$P(A \mid S_1 = 5) = \binom{5}{3} 0.8^3\, 0.2^2 + \binom{5}{4} 0.8^4\, 0.2 + 0.8^5$$