1. Let $X_1, X_2, \ldots, X_n$ be i.i.d random variables, where $X_i \sim Bernoulli(p)$. Define

\[
Y_1 = X_1 X_2,
Y_2 = X_2 X_3,
\vdots
Y_{n-1} = X_{n-1} X_n,
Y_n = X_n X_1.
\]

If $Y = Y_1 + Y_2 + \cdots + Y_n$, find

(a) $E[Y]$.
(b) $\text{Var}(Y)$.

\textbf{Solution:} Note that by symmetry:

\[
E[Y_1] = E[Y_2] = \cdots = E[Y_n]
E[Y_1] = E[X_1 X_2] = EX_1 EX_2 = p^2
\]

In particular, note that $Y_i \sim Bernoulli(p^2)$. Thus,

(a)

\[
EY = EY_1 + EY_2 + \cdots + EY_n
= np^2
\]

(b)

\[
\text{Var}(Y) = \sum_{i=1}^{n} \text{Var}(Y_i) + 2 \sum_{i<j} \text{Cov}(Y_i, Y_j)
\]

Note:

\[
\sum_{i=1}^{n} \text{Var}(Y_i) = n \text{Var}(Y_1) \quad \text{(By symmetry)}
= np^2 (1 - p^2)
\]

\[
\sum_{i<j} \text{Cov}(Y_i, Y_j) = \text{Cov}(Y_1, Y_2) + \text{Cov}(Y_2, Y_3) + \cdots + \text{Cov}(Y_{n-1}, Y_n) + \text{Cov}(Y_1, Y_n)
= n \text{Cov}(Y_1, Y_2)
\]

\[
\text{Cov}(Y_1, Y_2) = E[Y_1 Y_2] - E[Y_1] E[Y_2]
= E[X_1 X_2^2 X_3] - p^4
= E[X_1] E[X_3] E[X_2^2] - p^4
\]
Note

\[ E[X_2^2] = E[X_2] \]
\[ = p \]

\[ \text{Cov}(Y_1, Y_2) = p^3 - p^4 \]
\[ = p^3(1 - p) \]

\[ \text{Var}(Y) = np^2(1 - p^2) + 2np^3(1 - p) \]
\[ = np^2(1 - p)(1 + p + 2p) \]
\[ = np^2(1 - p)(1 + 3p) \]
2. Solve the following problems using the Central Limit Theorem (CLT):

(a) A binary communication makes an error on each bit with probability 0.15. Estimate the probability that more than 165 bit errors are made in 1000 bit transmissions. (Can you write down the exact expression for this probability? Given that you can easily write down the exact expression, why is the Central Limit Theorem useful here?)

(b) The number of messages arriving per second at a switch in a computer network is a Poisson random variable with mean 10 messages/second. Estimate the probability that more than 650 messages arrive in one minute.
Let \( Y = \sum_{i=1}^{1000} X_i \)

Where \( X_i = \begin{cases} 
1 & \text{bit i in error} \\
0 & \text{else}
\end{cases} \)

\( p(x_i=1) = p = 0.15 \)
\( p(x_i=0) = 1-p = 0.85 \)

\[ E [X_i] = p \cdot 1 + (1-p) \cdot 0 = p = 0.15 \]
\[ E [X_i^2] = p \cdot 1^2 + (1-p) \cdot 0^2 = p = 0.15 \]

\[ \Rightarrow \text{Var}(X_i) = p \cdot p^2 = 0.1275 \]

\[ P(Y > 165) = 1 - P\left( \sum_{i=1}^{1000} X_i \leq 165 \right) \]

Approximate by \( \chi^2 \) distribution:

\[ = 1 - P\left( \sum_{i=1}^{1000} X_i - 0.15 \cdot 1000 \leq \frac{165 - 0.15 \cdot 1000}{\sqrt{1000 \cdot 0.1275}} \right) \]

\[ = 1 - P\left( \frac{2}{1000} \leq 1.33 \right) \]

\[ = 1 - \Phi(1.33) \]

Exact: Use Bernoulli:

\[ \sum_{k=16}^{1000} \binom{1000}{k} 0.15^k 0.85^{1000-k} \]

Evaluation of large factorials can be challenging.

No insight on probability dependence on \( N, p \).
Let $X_i$: number of messages during second $i$

Want $P\left( \sum_{i=1}^{60} X_i > 650 \right)$

$$\mu = E[X_i] = 10$$
$$\sigma^2 = E[X_i^2] = 10$$

$$P\left( \sum_{i=1}^{60} X_i > 650 \right) = P\left( \sum_{i=1}^{60} X_i - 600 > 650 - 600 \right)$$

$$= P\left( \frac{\sum_{i=1}^{60} X_i - 600}{\sqrt{60(10)}} > \frac{50}{\sqrt{600}} \right)$$

$$= P\left( Z > 2.04 \right)$$

$$= 1 - P\left( Z \leq 2.04 \right)$$

By CLT:

$$\approx 1 - 0.0228$$
3. The amount of time needed for a certain machine to process a job is a random variable with mean $EX_i = 10$ minutes and $\text{Var}(X_i) = 2$ minutes$^2$. The time needed for different jobs are independent from each other. Using the CLT Find the probability that the machine processes less than or equal to 40 jobs in 7 hours.

Solution:

$$Y = X_1 + X_2 + \cdots + X_{40}$$

$EX_i = 10, \text{Var}(X_i) = 2$

$EY = 40 \times 10 = 400$

$\text{Var}(Y) = 40 \times 2 = 80$

$P(\text{Less than or equal to 40 jobs in 7 hours}) = P(Y > 7 \times 60)$

$= P(Y > 420)$

$= P\left(\frac{Y - 400}{\sqrt{80}} > \frac{420 - 400}{\sqrt{80}}\right)$

$\approx 1 - \Phi\left(\frac{20}{\sqrt{80}}\right) \approx 0.0127$
4. You have a fair coin. You toss the coin \(n\) times. Let \(X\) be the portion of times that you observe heads. How large \(n\) has to be so that you are 95% sure that \(0.45 \leq X \leq 0.55\)? In other words, how large \(n\) has to be so that

\[
P(0.45 \leq X \leq 0.55) \geq 0.95
\]

**Solution:**

\[
X = \frac{X_1 + X_2 + \cdots + X_n}{n} = \frac{Y}{n} \quad \text{where} \quad Y = X_1 + X_2 + \cdots + X_n
\]

\(X_i \sim \text{Bernoulli} \left( \frac{1}{2} \right)\)

\(EX_i = \frac{1}{2}\)

\(\text{Var}(X_i) = \frac{1}{4}\)

\(EY = \frac{n}{2}\)

\(\text{Var}(Y) = \frac{n}{4}\)

\[
P(0.45n \leq Y \leq 0.55n) = P \left( \frac{0.45n - 0.5n}{\sqrt{\frac{n}{2}}} \leq \frac{Y - 0.5n}{\sqrt{n}} \leq \frac{0.55n - 0.5n}{\sqrt{n}} \right)
\]

\[
\approx \Phi(0.1\sqrt{n}) - \Phi(-0.1\sqrt{n}) = 0.95
\]

\(2\Phi(0.1\sqrt{n}) - 1 = 0.95\)

\(\Phi(0.1\sqrt{n}) = 0.975\)

\(0.1\sqrt{n} \approx 1.96\)

\(n \geq 385\)
5. (You are learning estimation this week in class, but, in fact, there is little new if you understand the meaning of the conditional probability density function $f_{Y|X}(y|x)$. Here, you get to look at two popular such estimators, and you can do this problem before you see the estimation material in class.) 

Suppose that $X$ and $Y$ have joint probability density function 

$$f_{X,Y}(x,y) = \begin{cases} 2(x+y), & 0 \leq y \leq 1, 0 \leq x \leq y \\ 0, & \text{otherwise} \end{cases}$$

(a) Find $f_{Y|X}(y|x)$, writing your limit for $x$ between constants, and your limits for $y$ as a function of $x$.

(b) Suppose that you have measured $X = 0.5$. Find the maximum a posteriori (MAP) estimate of $Y$ given $X = 0.5$, which is defined as the value of $y$ such that $f_{Y|X}(y|0.5)$ is maximized; that is, the MAP estimator is $\hat{Y}_{MAP} = \text{argmax}_y f_{Y|X}(y|0.5)$.

(c) Suppose that you have measured $X = 0.5$. Find the minimum mean squared estimator (MMSE) estimate of $Y$ given $X = 0.5$, which is defined as $\hat{Y}_{MMSE} = E[Y|X = 0.5]$. 
(a) \[ f_x(x) = \int_x^1 2(x+y) \, dy = (2xy + y^2) \Big|_x^1 = (3x+1) - (2x^2 + x^2) = 1 + 2x - 3x^2 \]

\[ = \begin{cases} 
1 + 2x - 3x^2, & 0 \leq x \leq 1 \\
0, & \text{else} \end{cases} \]

\[ \int f_x(x) \, dx = 1 \checkmark \]

(b) \[ f_{\mathbf{x} \mid y=1,0 \leq y \leq 1} = \left\{ \begin{array}{ll}
\frac{2(x+y)}{1+3x+3y^2}, & 0 \leq x \leq 1 \\
0, & \text{else} \end{array} \right. \]

\[ \int f_{\mathbf{x} \mid y=1} \, dx = \begin{cases} 
\frac{1}{5} + \frac{8}{15} & 0 \leq y \leq 1 \\
0, & \text{else} \end{cases} \]

\[ \int f_{\mathbf{x} \mid y=1} \, dy = 1 \checkmark \]

(c) \[ E[\mathbf{y} \mid x=0.5] = \int_{0.5}^1 y (\frac{2}{5} + \frac{8}{15} y) \, dy = \left( \frac{2}{5} y^2 + \frac{8}{15} y^3 \right) \bigg|_{0.5}^1 = \frac{2}{5} + \frac{8}{15} - \left( \frac{1}{10} + \frac{4}{15} \right) = \frac{11}{15} - \frac{5}{30} = \frac{22}{30} \]