ECE 603 - Probability and Random Processes, Fall 2016

Homework #7

Due: December 7, 2016 (in class)

1. *Central Limit Theorem:*

   (a) A fair die is tossed 100 times:

   - Using Cheybshev's inequality, find a bound on the probability that the total number of spots (sum of faces that show) is between 316 and 384.
   - Using the Central Limit Theorem, estimate the probability that the total number of spots if between 316 and 384. Compare your answer to part (a).

   (b) A binary communication makes an error on each bit with probability 0.15. Estimate the probability that more than 180 bit errors are made in 1000 bit transmissions. (Can you write down the exact expression for this probability?)

   (c) The number of messages arriving per second at a switch in a computer network is a Poisson random variable with mean 10 messages/second. Estimate the probability that more than 650 messages arrive in one minute.

2. Let $X_0, X_1, X_2, \ldots$ be a discrete-time random process consisting of **independent** random variables, **each** with cumulative distribution function:

$$F_X(x) = \begin{cases} 0, & x \leq -1 \\ 1 - x^2, & -1 \leq x \leq 0 \\ 1, & x \geq 0 \end{cases}$$

   (a) Find $P(X_0 \geq -\frac{1}{2})$.

   (b) Find $E[X_0]$, the expected value of X_0.

   (c) Find $Var[X_0]$, the variance of X_0.

   (d) Let $U = \sum_{i=1}^{180} X_i$. Estimate $P(U \geq -115)$.

   Define a new random process $Y_1, Y_2, Y_3, \ldots$ such that: $Y_i = X_i - X_{i-1}$; that is, $Y_1 = X_1 - X_0$, $Y_2 = X_2 - X_1$, $Y_3 = X_3 - X_2, \ldots$.

   (e) Find $m_Y[n] = E[Y_n]$, the mean function of Y_n.

   (f) Find $R_Y[m,n] = E[Y_m Y_n]$, the autocorrelation function of Y_n. Is Y_n wide-sense stationary?

3. Let $Z(t) = Xt + Y$, where $X \sim N(0, 2)$ and $Y \sim N(1, 1)$, and X and Y are independent.

(a) Find the probability density function $f_{Z(t)}(x)$.

(b) Find the mean function $m_Z(t)$.

(c) Find the autocorrelation function $R_Z(t_1, t_2)$.

(d) Is $Z(t)$ wide-sense stationary (WSS)?

4. In a simple quadrature amplitude modulation (QAM) communication system, $X(t)$ is defined as:

$$X(t) = A\cos(2\pi f_c t) + B\sin(2\pi f_c t)$$

where f_c is the "carrier" frequency, and A and B are **independent** random variables with mean zero and variance σ^2. Assume all of the moments of A and B, except the first moment (mean), are non-zero.

(a) Find the mean function of $X(t)$.

(b) Find the autocorrelation function of $X(t)$.

(c) Is $X(t)$ wide-sense stationary?

(d) Is $X(t)$ strict-sense stationary? (*Hint: To show that it is not, try showing that $E[X^3(t)]$ depends on t.*).

5. Let T be a Gaussian random variable with mean 1 and variance 4. I observe the random process $X(t) = t - T$.

(a) Find the probability density function $f_{X(t)}(x)$ for all t.

(b) Find the mean function and autocorrelation function for $X(t)$.

(c) Determine whether $X(t)$ is a Gaussian random process.