1. Let \( Z(t) = X t + Y \), where \( X \) is a Gaussian random variable with mean 0 and variance 2, and \( Y \) is a Gaussian random variable with mean 1 and variance 1. Assume \( X \) and \( Y \) are independent.

   (a) Find the mean function \( m_Z(t) \).

   (b) Find the autocorrelation function \( R_Z(t_1, t_2) \).

   (c) Is \( Z(t) \) wide-sense stationary (WSS)?

2. In a simple quadrature amplitude modulation (QAM) communication system, \( X(t) \) is defined as:

   \[
   X(t) = A \cos(2\pi f_c t) + B \sin(2\pi f_c t)
   \]

   where \( f_c \) is the “carrier” frequency, and \( A \) and \( B \) are independent random variables with mean zero and variance \( \sigma^2 \). Assume all of the moments of \( A \) and \( B \), except the first moment (mean), are non-zero.

   (a) Find the mean function of \( X(t) \).

   (b) Find the autocorrelation function of \( X(t) \).

   (c) Is \( X(t) \) wide-sense stationary?
3. Suppose that $X(t)$ is a zero-mean wide-sense stationary random process with autocorrelation function $R_X(\tau) = \text{sinc}^2(\tau)$.

(a) Suppose that I run $X(t)$ through an ideal lowpass filter with unity gain and bandwidth $0.5$ Hz (Be sure you take into account the bandwidth!) as shown below:

![Ideal LPF](image)

Find:
- $S_Y(f)$, the power spectral density of $Y(t)$.
- $P_Y$, the power in $Y(t)$.

(b) Suppose that I run $X(t)$ through an ideal differentiator as shown below:

![Differentiator](image)

- $S_Y(f)$, the power spectral density of $Y(t)$.
- $P_Y$, the power in $Y(t)$.

4. Suppose that we have a message signal $M(t)$, which is a lowpass process of bandwidth $W$, that we transmit with DSB-SC:

$$X(t) = A_c M(t) \cos(2\pi f_c t)$$

The received signal is given by $R(t) = X(t) + W(t)$, where $W(t)$ is a zero-mean white noise process with power spectral density $S_W(f) = \frac{N_0}{2}$. Suppose that we use the standard receiver for DSB-SC in noise as given in class, but that the local oscillator at the receiver has a phase error; that is, the local oscillator is given by $\cos(2\pi f_c t + \phi)$.

(a) Assuming $\phi$ is fixed, derive an expression for the receiver output $Y(t)$ as a function of $M(t)$, $\phi$, $N_I(t)$, and $N_Q(t)$ (where $N_I(t)$ and $N_Q(t)$ are the in-phase and quadrature components, respectively, of the narrowband noise at the output of the BPF).

(b) Assuming that $\phi$ and $M(t)$ are fixed, find the expected value of $Y(t)$ as a function of $\phi$ and $M(t)$.

(c) Assuming that $\phi$ is fixed, find the output SNR as a function of $\phi$ and $P_m$, the power in the message.

(d) Assuming that $\phi$ is uniformly distributed between $-\frac{\pi}{4}$ and $\frac{\pi}{4}$, find the average output SNR.
5. Let $M(t)$ be a random process with power spectral density $S_M(f) = \frac{1}{2000}P(f/5000)$.

(a) Find the power in $M(t)$.

The message $M(t)$ is amplified and transmitted across a channel where noise is added. Suppose that the receiver desires to find $\frac{dM(t)}{dt}$, and thus the entire system looks as follows:

![System Diagram]

where $N(t)$ is a zero-mean white Gaussian noise process with power spectral density $S_N(f) = \frac{N_0}{2}$.

(b) Find the bandwidth $W$ of the (perfect) lowpass filter (LPF) that, while passing all of the signal, filters out as much noise as possible.

(c) Find the signal-to-noise ratio (SNR) at the receiver output in terms of $A$, $N_0$, and $W$ (i.e. use $W$ instead of your answer to (b) to avoid unwieldy algebra).

(d) Find the signal-to-noise ratio (SNR) at the receiver output in terms of $P_T$, $N_0$, and $W$ (i.e. convert your answer to (c) to be in terms of $P_T$ rather than $A$).

(e) Recognizing that the noise in the output is larger at higher frequencies because of the differentiator, you think of the following (clever?) architecture:

![Alternative System Diagram]

where

$$|H_2(f)|^2 = \begin{cases} \frac{3}{2} - \frac{|f|}{W}, & |f| < W \\ 0, & \text{else} \end{cases}$$

and $H_1(f) = 1/H_2(f)$. Find the receiver output SNR in terms of $A$, $N_0$, and $W$ (i.e. use $W$ instead of your answer to (b) to avoid unwieldy algebra).

(f) How much better (or worse) is the SNR of the system in (e) compared to the system in (d). (You can assume that the transmit power $P_T$ in terms of $A$ is approximately the same in both systems, since it is difficult to calculate it for the system in (e).)