ECE 603 - Probability and Random Processes, Fall 2016

Homework #6

Due: November 11, 2016 (in class)

1. Consider an experiment where I draw a piece of fruit from a shopping bag. The bag contains an Apple, Banana, Lime, Pear, and Orange. Since I draw them equally likely, I define the probability space $(S, \mathcal{A}, P)$ as:

$$S = \{\text{Apple, Banana, Lime, Pear, Orange}\}, \quad \mathcal{A} = \mathcal{P}_S \text{ (power set)}, \quad P(A) = \frac{|A|}{5},\ A \in \mathcal{A}$$

The fruit rots at different rates depending on the type. Using advanced science, I calculate how "edible" a fruit is on day n to arrive at the sequence of random variables $\{X_n\}$ defined by:

$$X_n(\omega) = \begin{cases} 1/n, & \omega = \text{Apple} \\ 2/n, & \omega = \text{Banana} \\ 3/n, & \omega = \text{Lime} \\ 4/n, & \omega = \text{Pear} \\ 5/n, & \omega = \text{Orange} \end{cases}$$

(a) Find $f_{X_1}(x)$ the probability density function of X_1.

(b) Does $\{X_n\}$ converge? If so, to what and in what ways?

2. [*Note: This problem will be done in class with different instructions: allowing you to use one type of convergence to imply another. I want you to see one problem solved under each set of instructions so that the distinction is clear.*] An experiment is defined by the probability space $(\Omega, \mathcal{A}, P)$, where $\Omega = [0, 1]$, $\mathcal{A}$ is the Borel σ-algebra restricted to $[0, 1]$, and $P(\cdot)$ is defined by $P((a, b)) = b - a$. Let $X_n(\omega) = \omega^n, \omega \in \Omega$. Determine whether or not the sequence $\{X_n\}$ converges (and to what) for each of the following cases. **Do the parts in order and be sure to justify each answer.**

(a) In distribution.

(b) In probability.

(c) In quadratic mean.

(d) Almost surely.

3. [This is an old exam problem.]

(a) Let the probability space $(\Omega, \mathcal{A}, P)$ be given by $\Omega = [0, 1]$, $\mathcal{A} = \mathcal{B}$ (restricted to $[0, 1]$, of course), and $P((a, b)) = b - a$. Let a sequence of random variables be defined by $X_n(\omega) = (\omega + \frac{1}{n})^2$.

- Find $f_{X_2|X_1}(x_2|x_1)$, the conditional probability density function of X_2 given X_1.

- Determine whether the sequence $\{X_n\}$ converges, and, if so, to what and in what ways? Consider almost sure convergence, mean square convergence, convergence in probability, and convergence in distribution. You can do them in any order that you want, and you can use one type of convergence to imply another when appropriate.

(b) Let the probability space $(\Omega, \mathcal{A}, P)$ be given by $\Omega = [-1/2, 1/2]$, $\mathcal{A} = \mathcal{B}$ (restricted to $[-1/2, 1/2]$, of course), and $P((a, b)) = b - a$. Let a sequence of random variables be defined by $X_n(\omega) = (-1)^n \omega$.

- Find $f_{X_1}(x)$, the probability density function of X_1
- Determine whether the sequence $\{X_n\}$ converges, and, if so, to what and in what ways? Consider almost sure convergence, mean square convergence, convergence in probability, and convergence in distribution. You can do them in any order that you want, and you can use one type of convergence to imply another when appropriate.

(c) Let X_n be integer random variables such that $P(X_n = 0) = 1 - 1/n$. $P(X_n = i) = 1/n^2, i = 1, 2 \ldots n$. Determine whether the sequence $\{X_n\}$ converges, and, if so, to what and in what ways? Consider mean square convergence, convergence in probability, and convergence in distribution. You can do them in any order that you want, and you can use one type of convergence to imply another when appropriate.

4. *Convergence Problems:*

(a) Let the probability space $(\Omega, \mathcal{A}, P)$ be given by $\Omega = [0, 1]$, $\mathcal{A} = \mathcal{B}$ (restricted to $[0, 1]$, of course), and $P((a, b)) = b - a$. Let $X_n(\omega) = \omega^3/\sqrt{n}$. Determine whether the sequence $\{X_n\}$ converges, and, if so, to what and in what ways? Consider almost sure convergence, mean square convergence, convergence in probability, and convergence in distribution. You can do them in any order that you want, and you can use one type of convergence to imply another when appropriate.

(b) Let the probability space $(\Omega, \mathcal{A}, P)$ be given by $\Omega = [0, 1]$, $\mathcal{A} = \mathcal{B}$ (restricted to $[0, 1]$, of course), and $P((a, b)) = b - a$. Let

$$X_n(\omega) = \begin{cases} \omega/n, & \omega \text{ irrational} \\ 1, & \omega \text{ rational} \end{cases}$$

Determine in what ways the sequence $\{X_n\}$ converges to $X = 0$. Consider almost sure convergence, mean square convergence, convergence in probability, and convergence in distribution. You can do them in any order that you want, and you can use one type of convergence to imply another when appropriate.

(c) Consider the probability space $(S, \mathcal{A}, P)$, with $S = [0, 1]$ and $\mathcal{A} = \mathcal{B}$ (restricted to $[0, 1]$,of course), and $P(\cdot)$ defined as follows:

$$P((a, b)) = \begin{cases} ((b-a) - \frac{1}{2}(b^2 - a^2)), & 0 \le a < b \le \frac{1}{4} \\ ((b-a) - \frac{1}{2}(b^2 - a^2)) & \frac{1}{4} \le a < b \le 1 \\ \frac{1}{2} + ((b-a) - \frac{1}{2}(b^2 - a^2)), & a < \frac{1}{4} < b \le 1 \end{cases}$$

Let

$$X_n(\omega) = \begin{cases} \omega/n, & \omega \text{ irrational} \\ 1, & \omega \text{ rational} \end{cases}$$

Determine in what ways the sequence $\{X_n\}$ converges to $X = 0$. Consider almost sure convergence, mean square convergence, convergence in probability, and convergence in distribution. You can do them in any order that you want, and you can use one type of convergence to imply another when appropriate.

(d) Let the probability space $(\Omega, \mathcal{A}, P)$ be given by $\Omega = [0, 1]$, $\mathcal{A} = \mathcal{B}$ (restricted to $[0, 1]$, of course), and $P((a, b)) = b - a$. Let

$$X_n(\omega) = \frac{\lfloor \omega \cdot n \rfloor}{n}$$

where $\lfloor x \rfloor$ is the largest integer less than x. Another way of stating $X_n(\omega)$: it is "rounding down" ω to the nearest c/n less than ω, where c is an integer. Determine whether the sequence $\{X_n\}$ converges, and, if so, to what and in what ways? Consider almost sure convergence, mean square convergence, convergence in probability, and convergence in distribution. You can do them in any order that you want, and you can use one type of convergence to imply another when appropriate.

(e) Let the probability space $(\Omega, \mathcal{A}, P)$ be given by $\Omega = [0, 1]$, $\mathcal{A} = \mathcal{B}$ (restricted to $[0, 1]$, of course), and $P((a, b)) = b - a$. Let $X_n(\omega) = \omega$, for $n = 0, 2, 4, 6, 8, \ldots$ (i.e. n even). Let $X_n(\omega) = 1 - \omega$, for $n = 1, 3, 5, 7, \ldots$ (i.e. n odd). Determine whether the sequence $\{X_n\}$ converges, and, if so, to what and in what ways? Consider almost sure convergence, mean square convergence, convergence in probability, and convergence in distribution. You can do them in any order that you want, and you can use one type of convergence to imply another when appropriate.