1. Let $X_1, X_2, X_3, \ldots$ be independent random variables, each with density function $f_X(x)$ with unknown mean $\mu$ and unknown variance $\sigma^2 = E[X^2] - \mu^2$. We estimate the sample mean as:

$$\hat{\mu} = \frac{1}{N} \sum_{i=1}^{N} X_i$$

After we have calculated the sample mean, the sample variance can be found as:

$$s^2 = \frac{1}{N-1} \sum_{i=1}^{N} (X_i - \hat{\mu})^2$$

(1)

In this problem, you will show why the latter equation is used instead of the more obvious choice.

(a) Show that:

$$\sum_{i=1}^{N} (X_i - \mu)^2 = \sum_{i=1}^{N} (X_i - \hat{\mu})^2 + N(\hat{\mu} - \mu)^2.$$  

(b) Use the result of part (a) to show that:

$$E\left[k \sum_{i=1}^{N} (X_i - \hat{\mu})^2\right] = k(N - 1)\sigma^2$$

(c) Use part (b) to show that $E[s^2] = \sigma^2$. When the expected value of an estimator is equal to what we are trying to estimate, it is called an unbiased estimator.

(d) Show that if $(N - 1)$ is replaced by $N$ in (1), the estimator becomes biased.

2. A fair die is tossed 100 times:

(a) Using Chebyshev’s inequality, find a bound on the probability that the total number of spots (sum of faces that show) is between 316 and 384.

(b) Using the Central Limit Theorem, estimate the probability that the total number of spots if between 316 and 384. Compare your answer to part (a).

3. Solve the following problems using the Central Limit Theorem:

(a) A binary communication makes an error on each bit with probability 0.15. Estimate the probability that more than 20 bit errors are made in 100 bit transmissions. (Can you write down the exact expression for this probability?)

(b) The number of messages arriving per second at a switch in a computer network is a Poisson random variable with mean 10 messages/second. Estimate the probability that more than 650 messages arrive in one minute.